Evolution of Population and a Resource with Maximin as Objective

Huilan Tian[1] and Robert D. Cairns[2*]

1. Division of Management

University of Toronto at Scarborough

email: helen.tian@utoronto.ca

2. Department of Economics

McGill University

email: robert.cairns@mcgill.ca

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Evolution of Population and a Resource with a Maximin Objective

Abstract. This paper investigates the properties of the dynamics of population and resource in a maximin model. Which of a continuum of steady states is approached depends on the initial conditions. For relatively large values of the resource stock, each steady state is conditionally stable in the saddlepoint sense; for small values of the resource stock, the approach path to a steady state is non-monotonic in the state space.

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1. Introduction

The literature on the simultaneous determination of optimal population and optimal stock levels for renewable natural resources has been dominated by utilitarian optimization. According to the utilitarian school, social welfare is a weighted sum of the welfare levels of individuals. If the time horizon is infinite, a solution often involves placing less weight on the welfare levels of generations in the distant future. Adding up utility levels across individuals and generations, however, whether with equal or unequal weights, is objectionable to some thinkers.

According to Rawls (1971), social welfare should be the welfare level of the least advantaged individual. Rawls's *maximin principle* has stimulated much research in economics. The criterion suggests trying to level the consumption of all individuals in society. Extending it to an intertemporal program, Solow (1974) shows that, if manufactured capital and an exhaustible resource are sufficiently substitutable, it is possible to sustain a constant level of consumption by depleting the resource stock toward zero and accumulating capital indefinitely. Hartwick (1977) proves what is now known as Hartwick's rule, that if the net value of investment is zero forever then a constant consumption stream is maintained forever.

While Hartwick's rule and its converse have been proved and generalized under ever more general conditions, less is known about properties of maximin paths. Solow's model constitutes the only example of a completely solved maximin problem. The present paper solves for the maximim path of an economy having a renewable resource and endogenous population growth. Unlike in models with a utililiarian objective where typically there is a unique steady state, we find a continuum of steady states. Which steady state will be approached depends on the initial conditions. For relatively large values of the resource stock, each steady state is conditionally stable in the saddlepoint sense. For small values of the resource stock, the approach path to a steady state is non-monotonic in the state space. A numerical simulation based on Brander and Taylor's (1998) stylization of Easter Island illustrates our conclusions.

2. Maximin Principle

Consider a dynamic system with n state variables s_i $(i = 1, 2, ...n)$ and m control variables c_j $(j = 1, 2, ..., m)$, where m can be greater than, equal to, or less than n. The state variables represent stocks such as natural resources, manufactured capital and population. The control variables represent the rates of harvesting, of investment, of consumption, etc. A set of n differential equations describes the dynamics of the system:

$$\dot{s}_i = f_i(s, c) \text{ for } i = 1, 2, ..., n. \tag{2.1}$$

Let $s = (s_1, s_2, ..., s_n)$ and $c = (c_1, c_2, ...c_m)$.

The utility level of the representative agent at time t is denoted by

$$u(t) = U(s(t), c(t)).$$

It is convenient to imagine that the representative agent at t is born at time t, enjoys the utility level $U(s(t), c(t))$, and immediately dies an instant after.

We assume that all functions are continuously differentiable.

The maximin objective is to achieve the highest level of utility that can be sustained for all time, or to

$$\max_{[c(t)]_{t=0}^{\infty}} \bar{u} \tag{2.2}$$

subject to the initial conditions $s_i(0) = s_{i0}$, the differential equations (2.1), and the maximin constraint,[1]

$$U(s(t), c(t)) \geq \bar{u}. \tag{2.3}$$

[1] Some problems may involve other inequality constraints. Abstracting from such constraints is justified if the only binding inequality constraint is (2.3).

We follow Cairns and Long (2001) in treating $\bar{u}$ as a control parameter. The Hamiltonian is

$$H(s, c, \pi) = \sum_i \pi_i f_i(s, c), \tag{2.4}$$

where π_i is the shadow price of s_i.[2] The corresponding Lagrangian is

$$\mathcal{L}(s, c, \pi, \mu, \bar{u}) = H + \mu \left[U(s, c) - \bar{u} \right].$$

We denote optimal levels by an asterisk. The following conditions are necessary:

- The vector c^* maximizes $H(s^*, c, \pi^*)$ subject to constraint (2.3):

$$\frac{\partial \mathcal{L}^*}{\partial c_j} = \sum_i \pi_i^* \frac{\partial f_i^*}{\partial c_j} + \mu^* \frac{\partial U^*}{\partial c_j} = 0,\ j = 1, ..., m. \tag{2.5}$$

- The multiplier μ^* satisfies the complementary slackness conditions:

$$\mu^*(t) \geq 0,\ U(s^*(t), c^*(t)) - \bar{u} \geq 0, \text{ and } \mu^*(t) \left[U(s^*(t), c^*(t)) - \bar{u} \right] = 0.$$

- The vector π^* satisfies the differential equations

[2] See also Leonard and Long (1992, esp.Theorem 7.11.1) and Cairns and Long (2002). The objective function is not an integral; heuristically speaking, the integrand is identically zero.

$$\dot{\pi}_k^* = -\frac{\partial \mathcal{L}^*}{\partial s_k} = -\sum_i \pi_i^* \frac{\partial f_i^*}{\partial s_k} - \mu^* \frac{\partial U^*}{\partial s_k},\ k = 1, ..., n. \tag{2.6}$$

- Along the optimal path,

$$\frac{d}{dt}\left[\mathcal{L}(s^*(t), c^*(t), \pi^*(t), \mu^*(t))\right] = \frac{\partial}{\partial t}\left[\mathcal{L}(s^*(t), c^*(t), \pi^*(t), \mu^*(t))\right] \tag{2.7}$$

  Since the Lagrangian does not contain t explicitly, $\frac{\partial}{\partial t}\left[\mathcal{L}(s^*(t), c^*(t), \pi^*(t)\right] = 0$; thus $\mathcal{L}(s^*(t), c^*(t), \pi^*(t))$ and *a fortiori* $H\left(s^*(t), c^*(t), \pi^*(t)\right)$ are constant.

- The optimal choice of the control parameter $\bar{u}$ maximizes the expression

$$\pi_0 \bar{u} + \int_0^\infty \mathcal{L}(s^*, c^*, \pi^*, \mu^*, \bar{u}) dt,$$

  where π_0 is a non-negative constant. Hence,

$$\pi_0 - \int_0^\infty \mu^*(t) dt = 0. \tag{2.8}$$

- The vector $(\pi_0, \pi_1, ..., \pi_n, \mu)$ is not identically zero.

- The following transversality condition has been proved by Michel (1982):

$$\lim_{t\to\infty} H(s^*(t), c^*(t), \pi^*(t)) = 0. \tag{2.9}$$

From conditions (2.7) and (2.9) we obtain

$$\sum_i \pi_i^*(t) f_i(s^*(t), c^*(t)) = 0. \tag{2.10}$$

Condition (2.10) is the converse of Hartwick's rule, that at all times on an efficient maximin path the value of net investment is zero.

Since $\mu(t) \geq 0$ and equations (2.5) and (2.6) are homogenous of degree one in all shadow prices, if $\mu(t)$ is not identically zero for all t then we can normalize the shadow prices by setting $\pi_0 = 1$. We assume that the following *regularity condition* holds:

$$\mu^*(t) > 0 \text{ for all } t.$$

The term regularity condition is borrowed from Burmeister and Hammond (1978). A maximin path is not regular if $\mu(t) = 0$ over some time interval. In such cases, there is redundant capacity at some times that cannot be "smoothed" over all time. Let $\rho(t) = -\dot{\mu}(t)/\mu(t)$ and $\psi_i(t) = \pi_i(t)/\mu(t)$.

We define a steady state of a maximin problem as a point (s_{ss}, c_{ss}) such that if the system ever attains that point, it will remain there forever.[3] For a steady state, it is consistent with all of the necessary conditions to let the $\psi_i(t)$ be constants. Substitution into condition (2.6) and division by μ yield that

$$(\dot{\mu}/\mu)\,\psi_k + \sum \psi_i \partial f_i/\partial s_k + \partial U/\partial s_k = 0, \tag{2.11}$$

so that $\rho = -\dot{\mu}/\mu$ is also constant in a steady state.

The shadow value $\mu(t)$ is a measure of the tightness of the maximin constraint (2.3). Condition (2.8) implies that, at least for large enough t, $\dot{\mu}(t) < 0$. If $\mu(t)$ decreases over time, a marginal relaxation of the constraint at a point of time t adds more to the objective function if t is nearer to the present. Conditions are not known for which $\dot{\mu}(t) < 0$ for all t.

In a steady state, since $\rho = -\dot{\mu}/\mu$ is constant, $\dot{\mu} < 0$ and $\rho > 0$. Also, by equation (2.11),

$$\rho_{ss}\psi_{kss} - \sum \psi_{iss}\frac{\partial f_i}{\partial s_{kss}} - \frac{\partial U}{\partial s_{kss}} = 0, \quad \text{for } k = 1, 2, ..., n \tag{2.12}$$

[3] In general, a maximin problem may or may not have a steady state, and even if it does the solution path may or may not converge to it.

We seek a point $(s_{ss}, c_{ss}, \psi_{ss}, \rho_{ss})$ in R^{2n+m+1} that satisfies equations (2.12) and the following:

$$\frac{\partial U}{\partial c_j} + \sum_{i=1}^{n} \psi_{iss} \frac{\partial f_i}{\partial c_j} = 0, \quad \text{for } j = 1, 2., .., m \tag{2.13}$$

$$f_k(s_{ss}, c_{ss}) = 0, \quad \text{for } k = 1, 2, ..., n \tag{2.14}$$

Since there are $2n+m$ equations and $2n+m+1$ unknowns, we expect a continuum of steady states.

3. A Maximim Model of Easter Island

In this section we adapt Brander and Taylor's (1998) model of Easter Island. There are two state variables, population or labor force $L(t)$ and a stock of a renewable resource, $S(t)$. A part of the labor force harvests the resource and the remainder produces a composite good. The average product of labor in harvesting is αS and in producing the composite good is unity. Let consumption per capita of the two goods be h and m. Then

$$m(t) = 1 - \frac{h(t)}{\alpha S(t)} \geq 0. \tag{3.1}$$

The representative individual has the utility function,

$$U(h, m) = h^{\beta} m^{1-\beta}, \text{ where } 0 < \beta < 1.$$

Let the natural birth rate, the natural death rate and the effect of consumption of the resource good (food) on the net rate of reproduction all be constant and be represented by b, d, and ϕ, respectively, and let $d > b$. Then the growth rate of population is

$$\dot{L}(t) = [b - d + \phi h(t)]\, L(t). \tag{3.2}$$

The growth rate of the resource stock is

$$\dot{S}(t) = rS(t) - r\,[S(t)]^2 / K - h(t)L(t) \tag{3.3}$$

Following Brander and Taylor, we also assume that the carrying capacity, K, is sufficiently great that

$$0 < \frac{d-b}{\alpha\beta\phi} < K.$$

This assumption ensures that there exist steady states with $\dot{\mu} < 0$, as is necessary by condition (2.8).

We now modify the model by introducing a social planner whose objective is to maximize the minimum level of utility, $\bar{u}$, over all times $t \geq 0$, subject to conditions (3.2) and (3.3) with initial values

$$L(0) = L_0 \text{ and } S(0) = S_0, \tag{3.4}$$

and to the maximin constraint, that for all $t \geq 0$,

$$U(h(t), m(t)) - \bar{u} = [h(t)]^{\beta} \left[1 - \frac{h(t)}{\alpha S(t)}\right]^{1-\beta} - \bar{u} \geq 0. \tag{3.5}$$

By equation (3.5), if $h(t) = \alpha S(t)$ then $m(t) = 0$, and hence $U(h(t), m(t)) = 0$, a level which is clearly not optimal. For any $S(t) > 0$, the planner must ensure that $h(t) < \alpha S(t)$.

The Lagrangian for this problem is

$$\mathcal{L}^E = \pi_1 \left[rS - \frac{rS^2}{K} - hL\right] + \pi_2 [b - d + \phi h] L + \mu \left[h^{\beta}\left(1 - \frac{h}{\alpha S}\right)^{1-\beta} - \bar{u}\right].$$

By our analysis above, the necessary conditions are:

$$\left(\frac{\beta \alpha S^* - h^*}{\alpha S^* h^*}\right)\left(\frac{\alpha S^* h^*}{\alpha S^* - h^*}\right)^{\beta} = (\psi_1^* - \phi \psi_2^*) L^*; \tag{3.6}$$

$$\dot{\psi}_1^* = \rho^*(t)\psi_1^* - r\psi_1^*\left[1 - \frac{2S^*}{K}\right] - (1-\beta)\frac{h^*}{\alpha S^{*2}}\left(\frac{\alpha S^* h^*}{\alpha S^* - h^*}\right)^{\beta}; \quad (3.7)$$

$$\dot{\psi}_2^* = \rho^*(t)\psi_2^* + \psi_1^* h^* - \psi_2^*\left[b - d + \phi h^*\right]; \quad (3.8)$$

$$\int_0^{\infty} \mu^*(t)dt = 1;$$

$$\mu^* \geq 0, h^{*\beta}\left[1 - \frac{h^*}{\alpha S^*}\right]^{1-\beta} - \bar{u} \geq 0, \mu^*\left[h^{*\beta}\left(1 - \frac{h^*}{\alpha S^*}\right)^{1-\beta} - \bar{u}\right] = 0;$$

$$\psi_1\left[rS - rS^2/K - hL\right] + \psi_2\left[b - d + \phi h\right]L = 0; \quad (3.9)$$

Furthermore, conditions (3.2), (3.3) and (3.4) are satisfied.

First we find the set of all potential steady-state, maximin points in the state space (S, L). It is convenient to define a larger set, $\mathcal{X}$, the set of *feasible stationary points* with strictly positive population. We exclude the steady states $(S, L) = (0, 0)$ and $(S, L) = (K, 0)$ because it is uninteresting to analyze sustainability with zero population. The set of steady-state maximin points is a proper subset of $\mathcal{X}$ because, as we have argued, in the steady state ρ must be positive. For example, the steady state approached under open access to the resource in Brander and Taylor's model is in $\mathcal{X}$ but is not a maximin point.

In the steady state, since $\dot{L} = 0$, equation (3.2) requires that $h = (d - b)/\phi =$

h_{ss}. Then we obtain from equation (3.3) with $\dot{S} = 0$ that

$$L = rS\left(1 - \frac{S}{K}\right)\left(\frac{\phi}{d-b}\right). \tag{3.10}$$

Equation (3.10) traces a concave curve in the space (S, L), with L tending to 0 as S tends to K. Since we require that $\alpha S > h_{ss}$, the set of feasible stationary points with postive population is

$$\mathcal{X} = \left\{(S, L) : K > S > \frac{d-b}{\alpha\phi} \text{ and } L = rS\left(1 - \frac{S}{K}\right)\left(\frac{\phi}{d-b}\right)\right\}.$$

For illustration, Figure 1 depicts the set with parameter values such that

$$\frac{d-b}{\alpha\phi} < \frac{K}{2} < \frac{d-b}{\alpha\beta\phi} < K,$$

as assumed by Brander and Taylor. Our results are essentially the same for other cases.

PLEASE PLACE FIGURE 2.1 HERE

Proposition 3.1. *There is a continuum of steady-state, maximin points* (S_{ss}, L_{ss})

with

$$\frac{d-b}{\alpha\beta\phi} < S_{ss} < K.$$

Corresponding to each value of S_{ss} there is a unique value of L_{ss}. As S_{ss} varies from $\dfrac{d-b}{\alpha\beta\phi}$ to K, the value of ρ_{ss} decreases and the maximin utility level increses.

Proof. Substituting the steady-state value $L_{ss} = rS_{ss}\left(1 - S_{ss}/K\right)/h_{ss}$ into equation (3.6) gives that

$$\left(\frac{\beta\alpha S_{ss} - h_{ss}}{\alpha S_{ss}}\right)\left(\frac{\alpha S_{ss} h_{ss}}{\alpha S_{ss} - h_{ss}}\right)^{\beta} = \left(\psi_{1ss} - \phi\psi_{2ss}\right)\left[rS_{ss}\left(1 - \frac{S_{ss}}{K}\right)\right]. \tag{3.11}$$

Setting $\dot{\psi}_i = 0$ $(i = 1, 2)$ in equations (3.7) and (3.8) yields that

$$\psi_{1ss}\left[\rho_{ss} - r + \frac{2rS_{ss}}{K}\right] = (1-\beta)\left(\frac{h_{ss}}{\alpha S_{ss}^2}\right)\left(\frac{\alpha S_{ss} h_{ss}}{\alpha S_{ss} - h_{ss}}\right)^{\beta} \tag{3.12}$$

and

$$\psi_{2ss} = -\frac{h_{ss}}{\rho_{ss}}\psi_{1ss}. \tag{3.13}$$

Equations (3.11), (3.12) and (3.13) determine $(\psi_{1ss}, \psi_{2ss}, \rho_{ss})$. Steady-state utility is

$$\bar{u} = h_{ss}^{\beta}\left(1 - \frac{h_{ss}}{\alpha S_{ss}}\right)^{1-\beta} = \left(\frac{d-b}{\phi}\right)\left[1 - \frac{d-b}{\alpha\phi S_{ss}}\right]^{1-\beta}, \tag{3.14}$$

and is greater for greater values of the steady-state stock S_{ss}. The shadow value of the resource stock, ψ_{1ss}, must be positive, since $\psi_{1ss} = \partial \bar{u} / \partial S_{ss}$. Therefore, we obtain using (3.13) that $\psi_{1ss} - \phi\psi_{2ss} > 0$. Since all the other factors in equation (3.11) are positive, $\beta\alpha S_{ss} - h_{ss}$ is positive. Since $h_{ss} = (d - b) / \phi$,

$$S_{ss} > \frac{d - b}{\alpha\beta\phi}.$$

Total differentiation of equation (3.12) shows that $d\rho_{ss}/dS_{ss} < 0$.

(PLEASE PLACE TABLE 2b HERE)

Our results on convergence are summarized in the following:

Proposition 3.2. *(i) For each steady-state pair* (S_{ss}, L_{ss})*, where* $S_{ss} \in (\frac{d-b}{\alpha\beta\phi}, K)$*, there exists a neighborhood of* (S_{ss}, L_{ss}) *in which there is a maximin path converging to* (S_{ss}, L_{ss})*. The path is monotone, with both* S *and* L *either increasing or decreasing.*

(ii) For any initial, stationary point (S_0, L_0)*, where* $S_0 \in (\frac{d-b}{\alpha\phi}, \frac{d-b}{\alpha\beta\phi})$*, there is no regular, maximin path. A path having two phases, of constant utility at different levels, can be defined as follows. In the first phase population falls and the resource stock rises until the pair* (S, L) *reaches a switching curve. Then the path enters the second phase; both population and the resource stock rise, approaching a steady*

state pair (S_{ss}, L_{ss}), *where* $S_{ss} \in (\frac{d-b}{\alpha\beta\phi}, K)$.

Our proof by linearization is found in Appendix I. Here we present highlights using algebra and graphs. Along a maximin path,

$$[h(t)]^{\beta}\left[1-\frac{h(t)}{\alpha S(t)}\right]^{1-\beta}=\bar{u}. \tag{3.15}$$

Taking total differentials of equation (3.15) yields that

$$(\beta\alpha S-h)\,dh+(1-\beta)\left(h^2/S\right)dS=0. \tag{3.16}$$

Clearly, along a maximin path, the shadow values are such that $\psi_1 \geq 0$ and $\psi_2 \leq 0$; also, for any maximin path leading to a feasible, stationary point with postive population, by equation (3.2) $L > 0$. Therefore, by equation (3.6), $\beta\alpha S - h \geq 0$; the line $h = \beta\alpha S$ in the space (S, h) forms a barrier to a maximin path. When $\beta\alpha S = h$, we must have $\psi_1 - \phi\psi_2 = 0$, so that $\psi_1 = 0$ and $\psi_2 = 0$.

When $h < \alpha\beta S$, $dh/dS < 0$ and when $h = \beta\alpha S$, $dh/dS = 0$. We can visualize these conditions by re-writing equation (3.15) as

$$S=\frac{h/\alpha}{1-(\bar{u}h^{-\beta})^{1/(1-\beta)}} \tag{3.17}$$

Equation (3.17) is depicted in Figure 3a. The curve NMR corresponds to a given level of utility, $\bar{u}$. At M the slope of the curve is zero. A higher utility level corresponds with a curve shifted up and to the right. (The slope is zero at point Q.) A lower utility level corresponds with a curve shifted down and to the left; the slope is zero at point Z. Each curve has slope zero at the line $h = \beta\alpha S$. Furthermore, when $h = \beta\alpha S$, by equation (3.16), $dS = 0$, and hence $\dot{S} = 0$. We have, then,

$$rS\left(1 - S/K\right) - hL = S\left[r\left(1 - S/K\right) - \beta\alpha L\right] = \dot{S} = 0, \text{ or}$$

$$L = \frac{r}{\alpha\beta}\left(1 - \frac{S}{K}\right). \tag{3.18}$$

The segment of this line between the points $\left(\frac{d-b}{\alpha\beta\phi}, \frac{r}{\alpha\beta}\left(1 - \frac{d-b}{\alpha\beta\phi K}\right)\right)$ and $(K, 0)$ is the lower limit of the maximin paths. Since $h > h_{ss} = (d-b)/\phi$, we have $\dot{L} = (b - d + \phi h) > 0$. Therefore, $dL/dS = \dot{L}/\dot{S} \to \infty$ as $h \to \alpha\beta S$ along a maximin path.

PLEASE PLACE FIGURES 2.3A AND 2.3B HERE.

Consider a representative steady-state point, $E_{ss} = (S_{ss}, L_{ss})$ in Figure 2.3b. The corresponding point in the space (h, S) in Figure 2.3a is E, where $h = h_{ss} =$

$(d-b)/\phi$ and $dS/dh < 0$. Near point E there is a constant-utility path leading to the steady state. (a) If $S_0 > S_{ss}$ the path is along NE. Along NE, $h < (d-b)/\phi$ and the population is decreasing. The corresponding path in Figure 2.3b is N^*E_{ss}. (b) If $S_0 < S_{ss}$ the path is ME; the corresponding path in Figure 2.3b is M^*E_{ss}. Since $h > (d-b)/\phi$, the population is increasing.

While we can extend the stable branch N^*E_{ss} for large value of the resource stock up to the value K, it is not possible to extend the stable branch M^*E_{ss} to lower resource stock sizes. In Figure 2.3a, in order to achieve the utility level $\bar{u}$, the resource stock must not be less than S_M, the stock size at point M. If we move backward in time from E_{ss}, the limit point of the constant-utility path $u = \bar{u}$ is M^*; at M^*, corresponding to M, the harvest level is $\alpha\beta S_M$. For a higher constant utility level, $\widehat{u}$, corresponding to the curve FQ in Figure 2.3a, the limit point is Q^* in Figure 2.3b, with Q^* to the right of M^*.

As is intuitively reasonable, the slope of the maximin trajectory M^*E_{ss} is strictly positive, because $dL/dS = \dot{L}/\dot{S}$ where $\dot{S}$ and $\dot{L}$ are both positive:

$$\dot{L} = (b - d + \phi h)L > (b - d + \phi\frac{d-b}{\phi})L = 0.$$

This is confirmed by equation (4.1) in the Appendix, with $h = \theta(S; \bar{u}) > h_{ss}$.

Now, we turn to other feasible, stationary points in $\mathcal{X}$ that are not maximin steady states because $S \leq (d-b)/(\alpha\beta\phi)$ and the value of ρ is not positive. Consider curve TPY in Figure 2.3a, where $S_Y > h_{ss}/\alpha = (d-b)/(\alpha\phi)$, as is required for utility to be positive. At stock level S_Y and population level

$$L_Y = \frac{\phi}{d-b} r S_Y \left[1 - \frac{S_Y}{K}\right]$$

we have a stationary point, in the sense that by setting $h = (d-b)/\phi$, we have $\dot{L} = 0 = \dot{S}$. The utility level is

$$\bar{u}_Y = [(d-b)/\phi]^{\beta} \left[1 - \frac{(d-b)/\phi}{\alpha S_Y}\right]^{1-\beta}$$

However, such a constant-utility stationary point is inefficient. Since $h > \beta\alpha S$, for the given value of S, $\partial u/\partial h < 0$. Utility can be increased by reducing the harvest until the line $h = \beta\alpha S$, at point Z, is reached; at Z utility is, say, $\bar{u}_Z > \bar{u}_Y$. In this case, $h < (d-b)/\phi$, and, by equation (3.2), $\dot{L} < 0$. By equation (3.16), utility can be maintained by reducing h and increasing S. Since (a) utility is positive at any stationary point, (b) utility is maintained, and (c) the curve of pairs (S, L) slopes strictly downward, eventually that curve will intersect the maximin paths.

Society can choose from among accessible maximin paths having a yet higher level of utility, by continuing to have utility level $\bar{u}_Z$ as higher paths are reached.. The maximin utility level for the whole time horizon is still $\bar{u}_Z$, because that is the utility level of the least fortunate generation. As we have argued above, at M^*, the slope of the path leading to E_{ss} is strictly positive. There is a kink at the junction point.

At this point, one may pause to ask the following question: from point M^*, instead of moving in the North-East direction as indicated by the line $M^* E_{ss}$, what prevents the planner from achieving a higher long-run utility path by moving East, i.e., along a horizontal line? The answer is that this plan requires setting $h = (d - b)/\phi$ to ensure $\dot{L} = 0$, and in the short-run there is some utility loss. In Figure 2.3a, the jump from $h_M = \alpha\beta S_M$ to $h = (d - b)/\phi$ entails a jump to a point below E, where utility is less than $\bar{u}$.

4. Discussion

The paths starting from stationary points with $S < (d-b)/(\alpha\beta\phi)$ are not regular. In an early phase the constant-utility level is lower than in the later phase. The reason is that the planner lacks the flexibility to shift the relatively plentiful resource stock which will be available in the later phase to the earlier phase to achieve an intermediate utility level for all time: there is only one control variable, h, while there are two state variables. Obviously, irregular paths can begin for initial pairs (S_0, L_0) not among the stationary points.

Therefore, there are two families of paths. One is a family of regular maximin paths converging to a steady state. For the pairs (S, L) lying on a maximin trajectory τ and some function υ_τ, utility is given by $\bar{u} = \upsilon_\tau(S, L)$, where $\partial\upsilon_\tau/\partial S > 0 > \partial\upsilon_\tau/\partial L$. The steady states are conditionally stable in the saddle-point sense. Along the other family of paths, utility is constant and then can pass into a second phase following a member of the first family with a higher, constant level of utility. These maximin paths are confined to the area given by $ABCD$ in figure 4. Outside this area, there is not a regular path.

Brander and Taylor's stylization had about forty people arriving at Easter Island, when the resource stock was equal to K. The maximin path for the model from the date of arrival would have had the population decrease to reach the stationary state. That the actual trajectory of the society was to increase to a population of around ten thousand is some indication of the divergence from a maximin solution implied by the parameters of the model. The steady state to which their economy was tending was at the point $(S, L) = \left(\frac{d-b}{\alpha\beta\phi}, \frac{r}{\alpha\beta}\left(1 - \frac{d-b}{\alpha\beta\phi K}\right)\right)$, a point just on the boundary of the set of maximin paths. If this point had been attained, the economy would have found it impossible to improve its utility by reducing the harvest and also to attain a higher level once the segment of the line (3.18) was reached.

Numerical simulations confirm our theoretical conclusions. Across steady states, a higher stock level entails a higher constant-utility level and a lower value of ρ. Since $\mu(t) = \rho e^{-\rho t}$ in a steady state, the latter result implies that for higher stock levels (and the corresponding, lower population levels) the tightness of the maximin constraint (2.3) is initially lower and declines more gently. In a sense, achieving a maximin solution is less onerous if there is a higher stock. Along the approach path to the steady state, the shadow price of the population is negative and ρ varies in the opposite direction to the population size. For example, along M^*E_{ss}, the value of $\rho(t)$ decreases toward its steady-state value.

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Appendix. Saddle-Point Property

Along the stable branch of the saddle point associated with the steady-state pair (S_{ss}, L_{ss}) and the maximin utility level $\bar{u}$, let $h = \theta(S;\bar{u})$. To find the slope of the stable branch at a point such as E_{ss} in Figure 2.3b, we first divide equation (3.2) by equation (3.3) to obtain

$$\frac{dL}{dS} = \frac{\dot{L}}{\dot{S}} = \frac{(b - d + \phi\theta(S;\bar{u}))\,L}{rS(1 - S/K) - \theta(S;u)\,L}.$$

At the steady state (S_{ss}, L_{ss}), we use L'Hôpital's rule and the fact that $b - d + \phi\theta(S;\bar{u}) = 0$ at (S_{ss}, L_{ss}) to get

$$\begin{aligned} \frac{dL}{dS} &= \frac{\frac{d}{dS}\left[(b - d + \phi\theta(S;\bar{u}))\,L\right]}{\frac{d}{dS}\left[rS(1 - S/K) - \theta(S;u)\,L\right]} \\ &= \frac{L\phi\frac{d\theta}{dS}}{r(1 - 2S/K) - \theta\frac{dL}{dS} - L\frac{d\theta}{dS}}, \text{ or} \end{aligned}$$

$$\left(\frac{dL}{dS}\right)^2 + \left(\frac{L}{h}\frac{d\theta}{dS} - \frac{r}{h}\left(1 - 2\frac{S}{K}\right)\right)\left(\frac{dL}{dS}\right) + \frac{L\phi}{h}\frac{d\theta}{dS}, \qquad (4.1)$$

a quadratic equation in dL/dS. Since $\frac{L\phi}{h}\frac{d\theta}{dS} < 0$ by equation (3.16), the two solutions are of opposite signs. We take $dL/dS > 0$ because our phase diagram, Figure 2.3a, indicates that the slope of the stable branch of the saddle point is positive.

Now let ε be a small, positive number and consider $S = S_{ss} + \varepsilon$. The corresponding value of L is $L_{ss} + \varepsilon \frac{dL}{dS}$. Starting from such a point, we run the system (3.2) and (3.3) backward in time (i.e., $t = 0, ..., -300$). Since we do not have an explicit functional form for $\theta(S; \bar{u})$, we replace the function $\theta(S; \bar{u})$ by its linear approximation,

$$\theta(S; \bar{u}) = \theta(S_{ss}; \bar{u}) + (S - S_{ss}) \frac{d\theta(S_{ss}; \bar{u})}{dS}.$$

Numerical simulations confirm that along the approach path to the steady state, $dL/dS > 0$ when $S_{ss} \in \left(\frac{d-b}{\alpha\beta\phi}, K \right)$.

Having solved for $\theta(S; \bar{u})$, we have a system of two differential equations conditional on $h = \theta(S; \bar{u})$. We analyze the stability of this system by linearizing the system (3.2) and (3.3):

$$\begin{bmatrix} \dot{L} \\ \dot{S} \end{bmatrix} = \begin{bmatrix} 0 & \phi L_{ss} \theta_S \\ -h^* & -L_{ss}\theta_S + r(1 - 2S_{ss}/K) \end{bmatrix} \begin{bmatrix} L - L_{ss} \\ S - S_{ss} \end{bmatrix}, \tag{4.2}$$

By equation (3.16),

$$\theta_S = -\frac{(1-\beta) h^{*2}}{(\alpha\beta S_{ss} - h^*) S_{ss}}.$$

Let the matrix on the RHS of equation (4.2) be represented by J, its determinant, $h^{*}\theta_S\phi L_{ss}$, by $\det J$ and its trace, $-L_{ss}\theta_S + r\left(1 - 2S_{ss}/K\right)$, by tr$J$. Its two characteristic roots are

$$\lambda_{1,2} = \frac{1}{2}\left[\mathrm{tr}J \pm \left((\mathrm{tr}J)^2 - 4\det J\right)^{\frac{1}{2}}\right].$$

The product of the roots is $\det J$ and the sum of the roots is tr$J$. If $\det J < 0$, there are a negative and a positive root, and the steady state has the saddle point property: there is a path converging to it. Since for all $S_{ss} \in \left(\frac{d-b}{\alpha\beta\phi}, K\right)$, $\theta_S < 0$, it follows that $\det J < 0$ in that interval. This proves stability in the saddlepoint sense.