One-sided Unsustainability Tests and NNP Measurement with Multiple Consumption Goods

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ONE-SIDED UNSUSTAINABILITY TESTS AND NNP MEASUREMENT WITH MULTIPLE CONSUMPTION GOODS

by

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Abstract: In an economy with multiple consumption goods (including environmental amenities) that uniquely maximizes the present value of utility with constant discounting, constant or falling augmented green net national product, or zero or negative augmented net investment, at any time implies that the economy is unsustainable then. "Augmented" means that time is included as a productive stock, which incorporates future exogenous technical progress and changes in world prices in a unified accounting framework. Examples are given of calculating accounting prices for multiple goods. The practical and philosophical rationale for testing sustainability in a present-value maximising, and therefore fully prescribed, development path is discussed.

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1. INTRODUCTION

There is now a well-established literature on measuring the sustainability of national economies. The generally acknowledged starting points are Pearce and Atkinson (1993) for eighteen countries worldwide and Pearce et al. (1993) for the U.K., and work has been continued for example by Atkinson et al. (1997), Hamilton and Clemens (1999) and Neumayer (1999). Despite this, there is still no formal, general, published statement and proof of the theoretical connection between sustainability, and measures of 'net investment', 'genuine saving' or the change in 'aggregate wealth'. Existing literature has one or more of the drawbacks of being informal (for example Pearce and Atkinson p. 104, Pearce et. al p. 42), unpublished (Pezzey 1994, Hamilton 1997), confined to a specific range of consumption and investment goods (Atkinson et al. pp. 62, 68), confined to the uninteresting and possibly non-existent case where an optimal development path already satisfies Hartwick's rule (Neumayer p. 151), or confined to exclude exogenous technical progress (most of the above). Asheim (1994), and later Vellinga and Withagen (1996, p511) and Aronsson and Lofgren (1998, p213), noted that positive or zero net investment at an instant does not imply sustainability then. Asheim further gave an example with positive net investment yet unsustainability during a finite interval. But none of these authors showed what *can* be implied from measuring net investment.

The main contribution here is to prove a pair of one-sided unsustainability tests in a perfectly competitive, present-value-maximising economy with multiple consumption goods (or 'extended consumption') and a constant utility discount rate. We show that at any time, negative or zero augmented net investment (including changes in non-marketed environmental resource stocks), or falling or constant augmented 'green' net national product (GNNP) measured with suitably indexed prices, implies that the level of instantaneous utility (wellbeing) is unsustainable then. (The meaning of ‘augmented’ will be explained shortly.) They are one-sided tests because they show only if an economy is unsustainable, but not if it is sustainable. The choice between net investment and GNNP change is enabled by a result in Asheim and Weitzman (2001), hereafter AW, which equates GNNP change with the interest on net investment, once an appropriate index of real prices has been determined.

We also give two extensions to, an observation on, and a correction to existing theory, whether published or not. The first extension is ‘augmentation’, which adds an extra term to net investment (and thus to GNNP) to account for all exogenous changes over time in the economy’s set of production possibilities, whether caused by (exogenous) technical progress, or by changing world interest rates and terms of trade. The theory thereby includes more specific results by Weitzman (1997) on technical progress, and Sefton and Weale (1996) and Vincent et al (1997) on trade. It also includes Asheim’s (1997) result on adjusting GNNP to incorporate capital gains resulting from exogenous changes in production possibilities, and thus to measure sustainable income when consumption is constant and the only determinant of utility.[2] The second extension is the inclusion of multiple ‘consumption’ goods, namely anything, including environmental amenity, that directly affects utility. With multiple consumption goods, sustainable income is hard to define without restrictive assumptions, but the results here are established without reference to sustainable income.

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2. Asheim (2000, p39) noted that despite a number of contrary early remarks in the literature on national income accounting, there is

> "no general result...on the relation between [green net national product] and [sustainable income] when neither consumption nor the interest rate is constant".

The observation is that the choice between net investment and GNNP change as equivalent tests of unsustainability is a practical choice between measures with different data requirements, and the equivalence can also be tested empirically. The correction is that ‘defensive’ costs, to be deducted from gross domestic product (GDP) in order to arrive at GNNP, include not just pollution abatement costs, but also the costs of discovering and extracting natural resources. A number of other small points are also made about the accounting details of measuring net investment.

The structure of the paper is as follows. Section 2 describes the economy, and establishes the one-sided sustainability tests and related results. Section 3 illustrates the tests for a specific economy with several realistic features: resource discovery and renewability, accumulation and abatement of pollution, exogenous technical progress and changing terms of trade. Section 4 uses an exact algebraic example to illustrate the sustainability tests in the context of two consumption goods. Section 5 considers how applicable the theory is likely to be in practice, and the apparent paradox of why sustainability measures should be of interest in an economy where present value maximisation already prescribes a unique development path, and so apparently leaves no role for sustainability concerns. Section 6 concludes.

2. TESTING FOR SUSTAINABILITY

2.1 Nature of the economy considered, and the time derivative of augmented GNNP

We consider a continuous-time, representative-agent, competitive economy as in AW, but with the important additions of treating time t, the cause of exogenous shifts in production possibilities, as a productive stock,[3] and of defining and investigating the economy's sustainability properties. Vector $\mathbf{C}(t)$ is an extended consumption bundle of everything, including environmental amenities, that influences the agent's current well-being, denoted by instantaneous utility $U(\mathbf{C}(t))$. The economy's endogenous stocks are denoted by a vector $\mathbf{K}(t)$ (of maybe different dimension to $\mathbf{C}(t)$), representing stocks of built capital, natural resources, environmental assets, human capital from education, and knowledge capital produced by research and development. All of these are endogenous because they can be influenced over time by choices in the economy. At any time, the combination of extended consumption $\mathbf{C}(t)$ and all stock changes $\dot{\mathbf{K}}(t)$ must lie within the smooth, convex production possibilities set $\Pi\{.\}$, which depends on $\mathbf{K}^{\dagger}$, our notation for endogenous stocks $\mathbf{K}$ together with the (exogenous) stock of time t:

$$[\mathbf{C}(t),\dot{\mathbf{K}}(t)] \in \Pi\{\mathbf{K}^{\dagger}(t)\} \quad \text{where} \quad \mathbf{K}^{\dagger} := (\mathbf{K},t) \qquad [1]$$

Any variable with the superscript † will be called *augmented* and will contain time as a stock, or some variable corresponding to time as a stock.

We assume that the economy at all times $t \geq 0$ maximizes the remaining present value (PV) of utility, which is taken to mean using a discount factor $\phi(s) = e^{-\rho s}$ with a constant discount rate $\rho > 0$. That is, it solves

$$\underset{\mathbf{C},\dot{\mathbf{K}}}{Max} \quad W(t) := \int_t^{\infty} U[\mathbf{C}(s)]e^{-\rho(s-t)}ds \quad \text{s.t.} \ [\mathbf{C}(s),\dot{\mathbf{K}}(s)] \in \Pi\{\mathbf{K}^{\dagger}(s)\}, \qquad [2]$$ [4]

It is implicit here that all externalities have already been somehow fully

3. This device is mentioned by Aronsson et al. (1997, p54) and developed by Pemberton and Ulph (2001).

4. *s* is used throughout as a variable time, e.g. within integrals.

internalized by environmental policies not represented in the model. We call this maximising path *PV-optimal*, or *optimal* for short. The assumption of PV-optimality is crucial but rather paradoxical in the context of sustainability, as discussed below in Section 5.

The current-value Hamiltonian of problem [2] is

$$H(\mathbf{C},\dot{\mathbf{K}}^{\dagger};\Psi^{\dagger}) := U(\mathbf{C}) + \Psi^{\dagger}.\dot{\mathbf{K}}^{\dagger}, \quad [3]$$

where the co-state vector $\Psi^{\dagger} = (\Psi,\Psi^{t})$ includes Ψ^{t} as the co-state variable of the stock of time. It is then trivial to show (so no proof is given) that the following augmented version of AW's equation (5) holds:

$$\dot{H}(t) = \rho\Psi^{\dagger}(t).\dot{\mathbf{K}}^{\dagger}(t) = \rho[H(t)-U(t)] \text{ by [3], for all } t. \quad [4]$$

We next define measures of prices and production, much as in AW. The shadow utility prices $(\partial U/\partial\mathbf{C})(t)$ of extended consumption goods and $\Psi^{\dagger}(t)$ of net investments, that come from the solution to [2], are unobservable. What can be observed in markets are nominal prices $\mathbf{p}(t)$ of $\mathbf{C}(t)$ and $\mathbf{q}(t)$ of $\mathbf{K}(t)$, denominated in money like dollars. There is also a nominal price of time, $q^{t}(t)$ which cannot be directly observed, but will be computed later in terms of observables. Nominal prices are proportional to utility prices, formally $\mathbf{p}(t) = [(\partial U/\partial\mathbf{C})(t)]/\lambda(t)$ and $\mathbf{q}^{\dagger}(t) := (\mathbf{q}(t),q^{t}(t)) = \Psi^{\dagger}(t)/\lambda(t)$, where $\lambda(t) > 0$ is the marginal utility of a dollar at time t. If we further deflate the nominal prices by some (yet to be defined) price index $\pi(t) > 0$, we get *real* prices $\mathbf{P}(t) := \mathbf{p}(t)/\pi(t)$ and $\mathbf{Q}^{\dagger}(t) = (\mathbf{Q}(t),Q^{t}(t)) := \mathbf{q}^{\dagger}(t)/\pi(t)$; hence

$$\mathbf{P}(t) = [(\partial U/\partial\mathbf{C})(t)]/\lambda(t)\pi(t) \quad \text{and} \quad \mathbf{Q}^{\dagger}(t) = \Psi^{\dagger}(t)/\lambda(t)\pi(t). \quad [5]$$

$Q^{t}(t) = \Psi^{t}(t)/\lambda(t)\pi(t)$ will be called the *value of time*, which measures the real value flow to the economy at t of time passing; later expressions derived for Q^{t} will give it a more intuitive meaning. To use AW's results, we choose the index $\pi(t)$ so that the Divisia property, which defines the sense in which the overall real price level is constant, is satisfied:

$$\pi(t) \text{ is s.t. } \dot{\mathbf{P}}(t).\mathbf{C}(t) = 0, \quad \text{for all } t. \qquad [6]$$

The real consumption discount factor $\Phi(t)$ is defined as the utility discount factor, times the marginal utility of money, times the price index:

$$\Phi(t) = e^{-\rho t}\lambda(t)\pi(t) \qquad [7]$$

The (real) consumption discount rate, which in a perfectly informed and maximising economy is the (real) *interest rate* $r(t)$, is then defined as

$$r(t) = -\dot{\Phi}(t)/\Phi(t) = \rho - \dot{\lambda}(t)/\lambda(t) - \dot{\pi}(t)/\pi(t). \qquad [8]$$

Green Net National Product (GNNP) $Y(t)$ is defined as:

$$Y(t) := \mathbf{P}(t).\mathbf{C}(t) + \mathbf{Q}(t).\dot{\mathbf{K}}(t). \qquad [9]$$

and we will call $\mathbf{Q}.\dot{\mathbf{K}}$, the value of net investments, just *net investment* (also called 'genuine saving' in Hamilton (1996) and his subsequent work).

Augmented GNNP is GNNP plus $Q^t(t)$, which using $\dot{\mathbf{K}}^\dagger = (\dot{\mathbf{K}},1)$ is

$$Y^\dagger(t) := Y + Q^t = \mathbf{P}.\mathbf{C} + \mathbf{Q}^\dagger.\dot{\mathbf{K}}^\dagger. \qquad [10]$$

Using the above assumptions and definitions, we can state the augmented version of AW's Proposition 3 as our first result. (The proof is omitted, as it just needs all relevant variables in AW's proof to be 'augmented.')

Proposition 1: The time derivative of augmented GNNP (after AW)

The time derivative of augmented GNNP $Y^\dagger(t)$ is always the real interest rate $r(t)$ times augmented net investment $\mathbf{Q}^\dagger(t).\dot{\mathbf{K}}^\dagger(t)$:

$$\text{For all } t, \ \dot{Y}^\dagger(t) = r(t)\mathbf{Q}^\dagger(t).\dot{\mathbf{K}}^\dagger(t) = r(t)[Y^\dagger(t) - \mathbf{P}(t).\mathbf{C}(t)] \ \text{ by [10].} \qquad [11]$$

(In fact this result holds even if the utility discount rate ρ is not constant. But our main result, Proposition 2 below, does require a constant rate, as already assumed in [2].)

2.2 The one-sided unsustainability tests

We start by defining the (maximum) *sustainable utility* $U_m(t)$ at any time, which depends on the economy's stocks at t, in the obvious way as

$$U_m(t) := \max U \text{ s.t. } U(\mathbf{C}(s)) \geq U \quad \text{for all } s \geq t. \qquad [12]$$

We then use as our sustainability definition:

$$\text{an economy is } \textit{sustainable} \text{ at time } t \;\Leftrightarrow\; U(t) \leq U_m(t). \qquad [13]$$

Using this definition ducks all debate about the notoriously prolific meanings of sustainability (see Pezzey 1992 for a historical collection). Contributing to that (often semantic) debate is not our aim here; we merely claim that [12] and [13] form a possible and fairly natural mathematical translation of the word 'sustainable' into the context of our representative-agent, smoothly substitutable model. For if the current level of wellbeing (utility) $U(t) > U_m(t)$, the economy is unable to sustain $U(t)$ from t forever after, since this would contradict the definition of $U_m(t)$; so wellbeing must fall below $U_m(t)$ at some finite time in the future.

Proposition 2: The one-sided unsustainability tests

Extra assumptions: The optimal utility path is *unique* and *non-constant*. (Hereafter, 'Extra assumptions' are what are required in addition to those made for the economy defined in Section 2.1.)

Result: At t, a non-rising augmented GNNP or non-positive augmented net investment means that the economy is unsustainable at t. That is:

$$\{\dot{Y}^{\dagger}(t) \le 0 \ \text{ or } \ \mathbf{Q}^{\dagger}(t).\dot{\mathbf{K}}^{\dagger}(t) \le 0\} \Rightarrow \{U(t) > U_m(t)\} \qquad [14]$$

or equivalently (from [10])

$$\{\dot{Y}(t) + \dot{Q}^t(t) \le 0 \ \text{ or } \ \mathbf{Q}(t).\dot{\mathbf{K}}(t) + Q^t(t) \le 0\} \Rightarrow \{U(t) > U_m(t)\} \qquad [15]$$

Proof: See Appendix.

Perhaps the most striking part of this result is that *zero* augmented GNNP change or net investment ensures *un*sustainability. Also, the reverse implications in [14] and [15] do not hold: positive augmented GNNP change or net investment does not imply sustainability, as already noted in the Introduction. However, in any economy where the welfare-maximising path is not unique, the first two inequalities in [14] and [15] must be strict ($\mathbf{Q}^{\dagger}.\dot{\mathbf{K}}^{\dagger} < 0$, etc) to be able to conclude that the economy is unsustainable. Also note that closely related to, but distinct from, Proposition 2 is the augmented form of Hartwick's rule: zero augmented net investment $\mathbf{Q}^{\dagger}.\dot{\mathbf{K}}^{\dagger}$ forever implies utility $U(t)$ is constant forever. (The distinction is that the sustainability test applies at a point in time, whereas, as stressed by Asheim 1994, p262, Hartwick's rule applies only over all time.) For by [5] and [4], $\rho\lambda\pi\mathbf{Q}^{\dagger}.\dot{\mathbf{K}}^{\dagger} = \rho\Psi^{\dagger}.\dot{\mathbf{K}}^{\dagger} = \dot{H} = \rho(H-U)$; hence $\{\mathbf{Q}^{\dagger}.\dot{\mathbf{K}}^{\dagger} = 0 \text{ forever}\} \Rightarrow \{H = U \text{ forever}\} \Rightarrow \{\dot{U} = \dot{H} = 0 \text{ forever}\}$.

To complete our main result, we show that Q^t, the 'value of time,' is the generalized present value of the partial time derivative of GNNP:

Proposition 3: The value of time

$$Q^t(t) = \int_t^\infty [\partial Y(s)/\partial s]\exp[-\int_t^s r(z)dz]ds$$
$$(= \int_t^\infty [\partial Y(s)/\partial s]e^{-r(s-t)}ds \quad \text{if } r \text{ is constant}) \qquad [16]$$

Proof: See Appendix.

Q^t is thus forward looking, and likely to be much harder to calculate than other elements of $\mathbf{Q}^\dagger$, which are based on current values only. Formula [16] takes on a more understandable form in Proposition 8 below, which shows how Q^t adds to GNNP because of future exogenous technical progress and future changes in world resource prices (and hence terms of trade).

2.3 Further results

Propositions 2 and 3 are the paper's main contribution. What follow here are two further results, Propositions 4 and 5, which relate more clearly to classic formal and informal results in the literature; and two more specialized cases, Propositions 6 and 7, of practical or theoretical interest.

Proposition 4, divided into cases with a varying or constant interest rate, slightly extends the generality of the classic Weitzman (1970, 1976) result on the present-value equivalence of GNNP. This itself has nothing to do with sustainability, but it has immediate implications for informal claims about the relationship between wealth and sustainability. We first need to define *real wealth* $\Theta(t)$ as the present value of extended consumption expenditures $\mathbf{P}.\mathbf{C}$ on the optimal path, using the real consumption discount factor:

$$\Theta(t) := \int_t^\infty \mathbf{P}(s).\mathbf{C}(s)[\Phi(s)/\Phi(t)]ds; \qquad [18]$$

and *wealth-equivalent income*, $Y_e(t)$, as the expenditure level which is the present value equivalent of expenditures on the optimal path:

$$\int_t^\infty Y_e(t)\Phi(s)ds := \int_t^\infty \mathbf{P}(s)\mathbf{.C}(s)\Phi(s)ds \;\;\Rightarrow\;\; \int_t^\infty Y_e(t)[\Phi(s)/\Phi(t)]ds = \Theta(t). \quad [19]$$

A further assumption, not needed for previous results, is needed for wealth-equivalent income Y_e to be well defined: the utility function $U(\mathbf{C})$ must be homothetic. Otherwise, the Divisia index $\pi(t)$ in [6] is path-dependent, and cannot be used to compare expenditures $\mathbf{P}.\mathbf{C}$ on different hypothetical development paths. This is not a problem when we assume a constant interest rate, since this effectively assumes a constant utility discount rate ρ, and a linear homogeneous utility function $U(.)$ which is a stronger restriction than homotheticity. (Note that Asheim (1997) developed PV-equivalence results for the cases of varying interest *and* utility discount rates.)

Proposition 4: The present-value equivalence of augmented GNNP (after Weitzman 1976 and Sefton and Weale 1996)

Extra assumption: The utility function $U(.)$ is homothetic.

Result (a): $Y^\dagger(t) = \int_t^\infty r(s)\mathbf{P}(s).\mathbf{C}(s)\exp[-\int_t^s r(z)dz]ds.$ [20]

Proof: [20] follows directly by integrating $\dot{Y}^\dagger(s) = r(s)[Y^\dagger(s)-\mathbf{P}(s).\mathbf{C}(s)]$ in Proposition 1 from time t to ∞, and assuming that the integral converges.

Extra assumptions: The utility function $U(.)$ is homothetic, and the interest rate r is constant, so $\Phi(s) = e^{-rs}$. Using [20], [18] amd [19] then gives:

Result (b): $Y^\dagger(t) = r\int_t^\infty \mathbf{P}(s).\mathbf{C}(s)e^{-r(s-t)}ds = r\Theta(t) = Y_e(t).$ [21]

The PV-equivalence result in Proposition 4(a) thus allows the terms of trade and the interest rate to vary (as did Sefton and Weale in their equation (8)), and also for technical progress (which they excluded). Proposition 4(b) shows that when the interest rate is constant, augmented GNNP $Y^\dagger$ is the same as wealth-equivalent income Y_e, and both can be seen as a return (at rate r) on wealth Θ. Two further comments then help to show the significance of these results.

First, in a *small, open economy,* where all investment prices are exogenous world prices, and with just one consumption good, wealth-equivalent income Y_e equals (maximum) sustainable consumption C_m (Asheim 2000, p38), which is defined analogously to (maximum) sustainable utility in [12]. The converse of the sustainability tests in Proposition 2 then do hold, that is, $\mathbf{Q}^\dagger.\dot{\mathbf{K}}^\dagger \gtrless 0 \Leftrightarrow C \lessgtr C_m$, and the comment in Asheim (1994) and Pezzey (1994) that there is no direct connection between GNNP and sustainability no longer applies. However, in the more realistic case of multiple consumption goods, this deduction fails, because the Divisia price index defining prices $\mathbf{Q}$ is generally different on the optimal and maximum sustainable paths. Also, the result obviously cannot work for the sum of all open economies (the world economy) where prices are no longer exogenous.

Second, in the case of constant interest rate, Propositions 4(b) and 2 together disprove the common view in the policy literature, dating at least from well-known but informally worded claims in Solow (1986, 1993) and Pearce et al. (1989), that non-declining wealth or 'aggregate capital' implies sustainability. For Proposition 4(b) means that $\dot{Y}^\dagger(t) = r\dot{\Theta}(t)$, and hence from Proposition 2 that momentarily *constant* wealth ($\dot{\Theta} = 0$) implies *un*sustainability.

Next, Proposition 5, proved by Weitzman (1997), restates his result on technical progress and income in terms of our value of time Q^t, and will be used to give Proposition 6. It actually applies to an open as well as closed economy, because the time dependence of the production possibilities set in [1] allows for the effect of exogenous changes in the terms of trade.

Proposition 5: The time premium (Weitzman 1997)

Extra assumption: The interest rate r is constant.

Result: The augmented GNNP (or the wealth-equivalent income) equals GNNP Y increased by a 'time premium' Q^t/Y:

$$Y^{\dagger}(t) = Y(t)[1 + Q^t(t)/Y(t)] = Y\,\{1 + \chi(t) \,/\, [r - \Gamma(t)]\} \quad [22]$$

where $\Gamma(t) := \int_t^{\infty} \dot{Y}(s)e^{-rs}ds \,/\, \int_t^{\infty} Y(s)e^{-rs}ds$ [23]

is the time-averaged overall growth rate of GNNP,

and $\chi(t) := \int_t^{\infty} [\partial Y(s)/\partial s]e^{-rs}ds \,/\, \int_t^{\infty} Y(s)e^{-rs}ds$ [24]

is the time-averaged growth in GNNP due to time alone.

The growth rate $\chi(t)$ in [24] is the result of any change in production possibilities caused by time alone, which is why we call $Q^t/Y = (Y^{\dagger}/Y) - 1$ the 'time premium,' rather than Weitzman's more specific 'technological progress premium.' A trivial corollary of Propositions 2 and 5 is then Proposition 6, which allows simpler sustainability tests than in Proposition 2 for the case when the rates of overall GNNP growth, and of GNNP growth due to time alone, are both constant.

Proposition 6: The one-sided sustainability tests with constant rates of GNNP growth and exogenous technical progress.

Extra assumptions: The optimal utility path is unique and non-constant. The interest rate r, instantaneous rate of overall GNNP growth, and instantaneous rate of GNNP growth due to time alone are all constant (with the last two being Γ and χ respectively).

Result: The assumptions and [22] $\Rightarrow Q^t(t) = Y(t)\chi/(r-\Gamma)$ and $\dot{Y}^\dagger/Y^\dagger = \dot{Y}/Y = \Gamma$. Proposition 2 then simplifies to these tests of unsustainability:

$$\{\dot{Y}(t) \leq 0 \text{ or } \mathbf{Q}(t).\dot{\mathbf{K}}(t) \leq -Y(t)\chi/(r-\Gamma)\} \Rightarrow \{U(t) > U_m(t)\}. \quad [25]$$

In practice, $\mathbf{Q}.\dot{\mathbf{K}} \leq 0$, i.e. net investment without the value of time premium Q^t, is the test used for most empirical measurements of sustainability. Weitzman (1997) and Vincent et al. (1997) are the only papers we know that include part of the time premium $\chi/(r-\Gamma)$ in their calculations. [25] shows that, if $\chi > 0$, sustainability is more likely than indicated by the $\mathbf{Q}.\dot{\mathbf{K}} \leq 0$ test that applies when $\chi = 0$. By contrast, the $\dot{Y}(t) \leq 0$ test is formally unaffected, and still means that any non-positive growth rate of GNNP implies unsustainability. Weitzman estimated $\chi/(r-\Gamma)$ to be about 0.4 for the USA, while the natural resource components of $\mathbf{Q}.\dot{\mathbf{K}}$ make up only about 0.03 of Y. However, Hamilton et al. (1998) suggested that some of $\chi/(r-\Gamma)$ could be endogenous technical progress; and some of it could also be changes in the terms of trade.

From a practical point of view, the second (net investment) conditions for unsustainability in each of Propositions 2 and 6 should be easier to compute than the first (GNNP change) conditions in the time-autonomous ($Q^t = 0$) case. This is because signing net investment ($\mathbf{Q}.\dot{\mathbf{K}}$), or finding its relative size $\mathbf{Q}.\dot{\mathbf{K}}/Y$, does not require the environmental valuation needed for any estimation of the extended consumption vector $\mathbf{C}$ or changes $\dot{\mathbf{P}}$ and $\dot{\mathbf{Q}}$ in real prices, whereas signing $\dot{Y}$ does require this. (If $Q^t \neq 0$, the comparison is not so simple.) Hanley et al. (1999) is the only empirical work we know of, done for Scotland in their case, which has tried to compare the GNNP growth ($\dot{Y}$) and net investment ($\mathbf{Q}.\dot{\mathbf{K}}$) measures. However, several trade terms that could be significant for a small, open economy like Scotland, as well as the more difficult value of time Q^t, were omitted from their analysis.

Finally, Proposition 7 shows how net investment can easily give a falsely optimistic message about sustainability on a development path before it reaches a single peak, after which the optimal path is permanently unsustainable. It is trivial to generalize this result to an economy with several peaks of utility, by redefining the 'initial' time to start somewhere on the upswing leading to the last peak.

Proposition 7: The false message of positive net investment in a single-peaked economy.

Extra assumptions: The optimal utility path is unique and non-constant.

Result: If net investment is initially positive and utility is single-peaked (that is, there is a time $T_P > 0$ such that $\dot{U}(t) \gtreqless 0$ for $0 \leq t \lesseqgtr T_P$), there is a finite time period when net investment is positive but utility is unsustainable.

Proof: This follows straightforwardly if tediously as a variant of the proof of Proposition 3 in Asheim (1994), and is available from the author.

3. A MORE SPECIFIC EXAMPLE

Here we use an example economy to give a better idea of how (augmented) GNNP change $\dot{Y}^\dagger$ or augmented net investment $\mathbf{Q}^\dagger.\dot{\mathbf{K}}^\dagger$ might be estimated in a real economy. The economy is less general than in Section 2, but it has a range of specific, familiar features like trade, and the costs of resource extraction and pollution abatement. Most of these occur somewhere in an existing model of GNNP accounting like Hartwick (1990), Maler (1991), Hamilton (1994, 1996), Vellinga and Withagen (1996) or Hamilton and Clemens (1999). Nevertheless, our economy reveals several new points: how to calculate explicitly the value of time; the reduced need for environmental valuation when estimating net investment rather than GNNP change; an accounting oversight regarding resource extraction and discovery costs; and a demonstration of how an existing result on resource trade forms part of the wider framework here.

The economy has n renewable and non-renewable domestic resource stocks, denoted by vector $\boldsymbol{S}(t)$. They are discovered at rate $\boldsymbol{D}(t)$, grow naturally at rate $\boldsymbol{G}(\boldsymbol{S}(t))$, and are extracted at rate $\boldsymbol{R}(t)$, so:

$$\dot{\boldsymbol{S}} = \boldsymbol{D} + \boldsymbol{G}(\boldsymbol{S}) - \boldsymbol{R}; \quad \boldsymbol{S}(0) = \boldsymbol{S}_0, \text{ given.} \qquad [26]$$

Two capital stocks in the domestic economy are productive capital $K(t)$ and abatement capital $K_a(t)$,[5] respectively increasing at investment rates $I(t)$ and $I_a(t)$ minus depreciations $\delta K(t)$ and $\delta K_a(t)$, $\delta > 0$ constant:

$$\dot{K} = I - \delta K; \qquad K(0) = K_0, \text{ given} \qquad [27]$$

$$\dot{K}_a = I_a - \delta K_a; \qquad K_a(0) = K_{a0}, \text{ given} \qquad [28]$$

Production $F(.)$ of a consumption-investment good depends positively on inputs of capital $K(t)$, domestic resource use (extraction $\boldsymbol{R}(t)$ minus net exports $\boldsymbol{R}_x(t)$) and time t (exogenous technical progress). Production plus net imports $M(t)$ is spent on produced consumption $C(t)$, investments $I(t)$ and $I_a(t)$, abatement current expenditure $a(t)$, discovery costs $V(\boldsymbol{D},\boldsymbol{S},t)$ with $V_{\boldsymbol{D}} > \boldsymbol{0}$, $V_{\boldsymbol{S}} < \boldsymbol{0}$, $V_t < 0$, and extraction costs $X(\boldsymbol{R},\boldsymbol{S},t)$ with $X_{\boldsymbol{D}} > \boldsymbol{0}$, $X_{\boldsymbol{S}} < \boldsymbol{0}$, $X_t < 0$:

$$F(K,\boldsymbol{R}-\boldsymbol{R}_x,t) + M = C + I + I_a + a + V(\boldsymbol{D},\boldsymbol{S},t) + X(\boldsymbol{R},\boldsymbol{S},t) \qquad [29]$$

The economy owns $K_f(t)$ foreign capital stock which earns a return at the

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5. Human capital and knowledge capital are ignored here, even though they are much bigger than abatement capital in real economies, because we have no new accounting points to make about these capitals.

exogenous world interest rate $r(t)$; and its net resource exports $\boldsymbol{R}_x(t)$ are sold at exogenous world prices $\boldsymbol{Q}^x(t)$ (x denotes exports, not extraction cost). So the change in foreign capital stock, with only exogenous dependences on time t shown, is:[6]

$$\dot{K}_f = r(t)K_f + \boldsymbol{Q}^x(t).\boldsymbol{R}_x - M; \quad K_f(0) = K_{f0}, \text{ given.} \qquad [30]$$

Importantly in view of our focus on multiple consumption goods, instantaneous utility U depends on both produced consumption $C(t)$ and environmental quality $B(t)$, so the extended consumption vector is $\mathbf{C} = (C,B)$:

$$U(t) = U[C(t),B(t)], \quad U_C,\, U_B > 0. \qquad [31]$$

Hence there are real prices of produced consumption, P^C, and environmental quality, P^B, in terms of an index of extended consumption. Environmental quality $B(t)$ in this particular model is some pristine level B_0, minus ε^B times the quality lost from flow pollution $E^B(t)$ (which depends on output, abatement expenditure, abatement capital and time), and minus ε^Ω times the drop in some environmental absorption capacity $\Omega(t)$ below its pristine, pre-industrial level Ω_1:

$$B = B_0 - \varepsilon^B E^B[F(K,\boldsymbol{R}-\boldsymbol{R}_x,t),a,K_a,t] - \varepsilon^\Omega(\Omega_1-\Omega), \quad \varepsilon^B,\, \varepsilon^\Omega > 0, \qquad [32]$$

and we denote:

$$b(t) := 1 \,/\, (\partial B/\partial a) \qquad [33]$$

as the marginal cost of improving environmental quality by abating emissions. (We model absorption capacity rather than cumulative pollution so all stocks in the model are goods, and so can satisfy the ‘free disposal’

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6. There is no distinction here between private or government ownership of foreign capital (or debt). For a large open economy, the interest rate r would depend also on the level of capital K_f, and the resource price $\boldsymbol{Q}^x$ would depend also on net exports $\boldsymbol{R}_x$.

assumption of Asheim (1997). We ignore the complication that total emissions of the most important cumulative pollutants, greenhouse gases in the global atmosphere, can be controlled only by the global economy, not by our example, open economy.) Finally, absorption capacity $\Omega(t)$ rises at emissions assimilation rate $\gamma(\Omega) > 0$, $\gamma' < 0$, and falls with emissions E^{Ω} which increase with domestic resource use $\boldsymbol{R}$-$\boldsymbol{R}_x$. Emissions can be abated only by reducing $\boldsymbol{R}$-$\boldsymbol{R}_x$:

$$\dot{\Omega} = \gamma(\Omega) - E^{\Omega}(\boldsymbol{R}\text{-}\boldsymbol{R}_x); \quad \Omega(0) = \Omega_0, \text{ given; } \Omega(-\infty) = \Omega_1 > \Omega_0. \qquad [34]$$

All functional forms are assumed to be as smooth and convex as is needed for generalized present value $W(t)$ in [2] to converge, for partial derivatives below with respect to extraction rates and time (denoted by subscripts $\boldsymbol{R}$ and s respectively) to exist, and for the maximising solution to exist, be unique, and be attained. As before, this means that optimal environmental policies to internalize all externalities are already, invisibly, in place. We then have:

Proposition 8: A detailed formula for augmented GNNP in a specific case

Result: $$Y^{\dagger} = P^C \{ C + bB + \dot{K} + \dot{K}_a + \dot{K}_f + (bB_{\boldsymbol{R}}+F_{\boldsymbol{R}}-\boldsymbol{Q}^x)_i\dot{\Omega}/(E^{\Omega}{}_{\boldsymbol{R}})_i + (\boldsymbol{Q}^x-X_{\boldsymbol{R}}).\dot{\boldsymbol{S}} \} + Q^t, \quad \text{where} \qquad [35]$$

$$Q^t(t) := \int_t^{\infty} P^C(s) \{bB_s+F_s-V_s-X_s + \dot{r}K_f + \dot{\boldsymbol{Q}}^x.\boldsymbol{R}_x\}(s) \exp[-\int_t^s r(z)dz]\, ds, \qquad [36]$$

and as part of the conditions for efficient resource use, all the $(bB_{\boldsymbol{R}}+F_{\boldsymbol{R}}-\boldsymbol{Q}^x)_i/(E^{\Omega}{}_{\boldsymbol{R}})_i$ are equal for $i = 1,...,n$, so any i will do.

Proof: See Appendix.

GNNP $Y^{\dagger}$ is thus the value sum of extended consumption, $C + bB$; the changes in the three capital stocks, $\dot{K} + \dot{K}_a + \dot{K}_f$, where the price is the same as for produced consumption; the change in the absorption capacity, $\dot{\Omega}$, valued at a price $(bB_{\boldsymbol{R}}+F_{\boldsymbol{R}}-\boldsymbol{Q}^x)_i/(E^{\Omega}{}_{\boldsymbol{R}})_i$ which reflects the various roles in the economy played by the resource flow $\boldsymbol{R}$; and the change in the resource stocks, $\dot{\boldsymbol{S}}$, valued at world prices $\boldsymbol{Q}^x$ minus their marginal extraction costs $X_{\boldsymbol{R}}$; plus the value of time Q^t. Q^t is in turn the discounted present value of the various sources of exogenous technical progress, as represented by the pure time derivatives $bB_t+F_t-V_t-X_t$ plus the 'capital gains' from exogenously changing world prices. These gains, a specific example of the more general analysis in Asheim (1996), are here the change in interest rate $\dot{r}$ times the economy's foreign capital K_f, and the changes in world resources prices $\dot{\boldsymbol{Q}}^x$ times the economy's resource exports $\boldsymbol{R}_x$ (not total extractions $\boldsymbol{R}$).

Four other points are worth noting as implications of Proposition 8:

(a) If there is no stock pollution ($\Omega = 0$), and no technical progress in abatement ($B_t = 0$), then as noted after Proposition 6, there is a practical difference between the two sustainability tests. The $\dot{Y}^{\dagger}$ test requires environmental evaluation, but the $\mathbf{Q}^{\dagger}.\dot{\mathbf{K}}^{\dagger}$ $(= P^C[\dot{K}+\dot{K}_a+\dot{K}_f+(\boldsymbol{Q}^x-X_{\boldsymbol{R}}).\dot{\boldsymbol{S}}])$ test does not.

(b) Since [27]-[30] imply

$$C + \dot{K} + \dot{K}_a + \dot{K}_f = C + I + I_a - \delta(K+K_a) + rK_f + \boldsymbol{Q}^x.\boldsymbol{R}_x - M$$
$$= F - a - V - X - \delta(K+K_a) + rK_f + \boldsymbol{Q}^x.\boldsymbol{R}_x, \qquad [37]$$

among the adjustments that must be made to F (i.e. gross domestic product, GDP) to arrive at augmented GNNP are deductions of total discovery and extraction costs (V and X). This accords with the intuition that, other things being equal, having to spend more on discovering and extracting a given amount of resources adds nothing to current or future wellbeing. It has sometimes been overlooked, for example by Hartwick (1990, p294) and Hamilton (1994, p159).

(c) The problem of translating results from utility to consumption units, bypassed in Hartwick (1990) with an approximate 'linearisation of the Hamiltonian,' is here transformed to the problem of finding how $P^C(t)$, the real price of produced consumption, changes over time. This must be inferred using $P^B = bP^C$ from the Appendix, and the Divisia property [6]: $\dot{\mathbf{P}}.\mathbf{C} = \dot{P}^C C + \dot{P}^B B = \dot{P}^C C + (b\dot{P}^C + \dot{b}P^C)B = 0$

$$\Rightarrow \ \dot{P}^C/P^C = -\dot{b}B/(C+bB). \qquad [38]$$

The problem is transformed rather than solved, because of the difficulties of calculating both marginal abatement cost $b(t)$ and environmental quality $B(t)$. However, we now have a precise formula [38] to aim at, rather than an unknown linearisation error.

(d) With no technical progress, $B_t = F_t = V_t = X_t = 0$, and Q^t becomes

$$Q^t = \int_t^\infty P^C(s)[\dot{r}(s)K_f(s) + \dot{\boldsymbol{Q}}^x(s).\boldsymbol{R}_x(s)] \exp[-\int_t^s r(z)dz]\, ds, \qquad [39]$$

which makes [35] similar to the open economy results of Sefton and Weale (1996, Section 4). If we also exclude domestic capital K, environmental quality B (hence P^C can be set to 1), abatement capital K_a, domestic production F, domestic resource use $\boldsymbol{R}-\boldsymbol{R}_x$ (so $\boldsymbol{R}_x = \boldsymbol{R}$), discoveries $\boldsymbol{D}$ and resource growth $\boldsymbol{G}$, and assume a constant interest rate r, [35] then becomes

$$Y^\dagger = C + \dot{K}_f - (\boldsymbol{Q}^x - X_{\boldsymbol{R}}).\boldsymbol{R} + \int_t^\infty \dot{\boldsymbol{Q}}^x(s).\boldsymbol{R}(s)e^{-r(s-t)}ds, \qquad [40]$$

Using the augmented Hartwick rule established after Proposition 2, this gives the main result (8) of Vincent et al. (1997) as a special case of our more general results.

4. AN EXACT ALGEBRAIC EXAMPLE

The concepts of Q^t, the value of time, and P^C, the price of produced consumption, are unfamiliar enough for the following exact example economy to be a useful illustration. It reveals some unfamiliar concepts such as a price of produced consumption different from 1, and an interest rate not equal to the marginal product of capital. The economy is closed, with a non-renewable resource stock S, $S(0) = S_0 > 0$, and depletion $R = -\dot{S}$. The only expenditures are investment $\dot{K}$, produced consumption C and abatement current spending a. Production is Cobb-Douglas in capital K, $K(0) = K_0 > 0$, and resource flow R, with exogenous technical progress at rate ν:

$$F(K,R,t) = K^{\alpha}R^{\beta}e^{\nu t} = \dot{K} + C + a, \quad 0 < \alpha,\ \beta < \alpha+\beta \leq 1; \quad \nu > 0. \qquad [41]$$

Utility U and environmental quality B are given by

$$U = \alpha \ln(C) + \varepsilon \ln(B), \quad \text{where } \varepsilon > 0 \text{ and } B = a/R. \qquad [42]$$

The functional form of [42] is selected deliberately so that a balanced growth path (which proves to be optimal), with abatement a growing at the same rate as C, and resource flow R inevitably declining towards zero, will have environmental quality a/R growing faster than produced consumption. So produced consumption C becomes more scarce over time relative to an index of extended consumption **C** and its price P^C rises, while environmental quality becomes relatively less scarce and its price P^B falls.

If the utility discount factor is $\phi(t) = e^{-\rho t}$ as before, and the parameters happen to obey

$$(\rho+\omega)K_0^{1-\alpha} = \alpha S_0^{\beta} \quad \text{where } \omega := (\nu-\beta\rho)/(1-\alpha), \qquad [43]$$

then the economy’s optimal path is given (proofs are available from the author) at all times by the balanced growth forms

$$K(t) = K_0 e^{\omega t}, \quad C(t) = \alpha a(t)/\varepsilon = \{[\rho+(1-\alpha)\omega]/(\alpha+\varepsilon)\}K(t), \qquad [44]$$

$$R(t) = \rho S(t) = \rho S_0 e^{-\rho t}, \qquad [45]$$

and $$B(t) = a(t)/R(t) = \{[\rho+(1-\alpha)\omega]/(\alpha+\varepsilon)\}\ (\varepsilon/\alpha\rho)\ (K_0/S_0)e^{(\rho+\omega)t}. \qquad [46]$$

The marginal abatement cost is $b = 1/(\partial B/\partial a) = R$, hence

$$bB = a = (\varepsilon/\alpha)C. \qquad [47]$$

Combining [38], [44] and [46], [47] and [38], the produced consumption price P^C rises at rate

$$\eta := \dot{P}^C/P^C = -(\dot{b}/b)\ /\ (1+C/bB) = \rho\ /\ (1+\alpha/\varepsilon) > 0, \qquad [48]$$

while the price of environmental quality P^B falls:

$$\dot{P}^B/P^B = \dot{b}/b + \dot{P}^C/P^C = -\rho/(1+\varepsilon/\alpha) < 0. \qquad [49]$$

The rate of interest r is the marginal product of capital ($F_K = \rho + \omega$) *plus* the rate of growth η of the produced consumption price;

$$r = \rho + \omega + \eta; \qquad [50]$$

the value of time [16] is

$$Q^t = \{\nu(\alpha+\varepsilon)(\rho+\omega)\ /\ \rho\alpha[\rho+(1-\alpha)\omega]\}\ P^C C; \qquad [51]$$

and wealth-equivalent income [19] is

$$Y_e = \{\varepsilon/\alpha + (\alpha+\varepsilon)(\rho+\omega)/\alpha\rho\}\ P^C C. \qquad [52]$$

Since the interest rate is constant, from [10], [21] and [52] we can then calculate (non-augmented) GNNP:

$$Y = Y_e - Q^t = \{\varepsilon/\alpha + (1-\beta)(\alpha+\varepsilon)(\rho+\omega)/\alpha[\rho+(1-\alpha)\omega]\}\ P^C C. \qquad [53]$$

and this result is (as required) the same as the appropriate ‘consumption plus net investment’ definition in [35], namely $Y = P^C(C+bB) + Q^K\dot{K} - Q^S R$.

In a balanced growth economy, sustainability means a positive growth rate of utility, which is

$$\dot{U} = \alpha\dot{C}/C + \varepsilon\dot{B}/B = (\alpha+\varepsilon)\rho\ [\omega/\rho + \varepsilon/(\alpha+\varepsilon)], \qquad [55]$$

Augmented net investment is

$$\mathbf{Q}^{\dagger}.\dot{\mathbf{K}}^{\dagger} = Q^K\dot{K} - Q^S R + Q^t = (1+\varepsilon/\alpha)\ [\omega/\rho + \varepsilon/(\alpha+\varepsilon)]P^C C; \qquad [56]$$

while from using [43] and [48] with either [53], or [52] and [21], the growth rate of GNNP and augmented GNNP is

$$\dot{Y}/Y = \dot{Y}^{\dagger}/Y^{\dagger} = \dot{P}^C/P^C + \dot{F}/F = \eta + \omega = \rho\ [\omega/\rho + \varepsilon/(\alpha+\varepsilon)]. \qquad [57]$$

The signs of [56] and [57] are the same as the sign of $\dot{U}$ in [55], as required by the one-sided unsustainability tests in Proposition 2.

5. THE PRACTICAL AND PHILOSOPHICAL BASIS OF TESTING FOR SUSTAINABILITY WHILE ASSUMING PV-OPTIMALITY

Despite its mathematical sophistication, the above neoclassical approach to sustainability contains some key approximations and an apparent paradox, which may limit its use as a practical tool for policy makers. The approximations arise as follows. Firstly, because of significant externalities and other departures from competitiveness, *current* prices and quantities observed in the market, including those estimated with non-market valuation techniques for the externalities, are significantly different from *optimal* prices that would be observed after policy intervention shifted development to the (PV-)optimal path defined by [2]. Secondly, suppose the (positive) definition [13] of sustainability is taken to imply a (normative) goal that development should be sustainable, as often found in current policy statements. Suppose also that the sustainability goal imposes a binding constraint on optimality. Then the *sustainability* prices and quantities, that would apply after further intervention achieves the constraint (presumably with minimum loss of PV), would be different again. The neoclassical approach to sustainability set out above[7] thus:

(i) Assumes that currently observed prices and quantities are adequate approximations of optimal prices and quantities;

(ii) Proves a theoretical inequality relationship (Proposition 2) between measures with optimal prices and quantities, and sustainability;

(iii) Assumes, although sometimes implicitly, that the goal of policy intervention is PV-maximisation, but subject to a sustainability constraint or modified by a public sustainability concern.

Assumption (iii) contains an apparent paradox, which is usually hidden and hence not discussed, since most literature focuses on measuring rather than achieving sustainability. Why should the government be interested in sustainability, and use intervention if necessary to achieve it, if private agents merely seek to maximize PV? For consumers to maximise PV has generally nothing to do with sustainability. It gives a complete and unique prescription for the time paths of every decision that ever has to be made in the economy; and because of the constancy of the discount rate ρ used to derive our sustainability tests, PV-maximisation can cause sustainability problems (Dasgupta and Heal 1979). Indeed, individuals must in fact believe there will be no policy intervention in favor of sustainability, or else they would modify their plans for the future, causing prices today not to be PV-optimal. So there is no apparent motive for using Proposition 2 to

7. This is an important caveat, because other writers may choose a different but still neoclassical approach. For example, Asheim (1997, 2000) included non-constant utility discounting; and Asheim et al. (2001) used an intergenerational equity axiom to find a rigorous justification of sustainability as a side constraint on maximising PV.

measure sustainability on a PV-optimal path, or for using intervention if unsustainability is thereby found.[8]

The resolution of this paradox has to lie in some kind of split between private and public concerns about the far future. We must assume that the individual chooses his or her own actions to maximize some form of present value, but votes for a government which applies a sustainability concern, both by measuring sustainability, and taking action to achieve it if necessary. People are thus in some sense schizophrenic, treating private economic decisions as the domain of Economic Man, and governmental decisions as the domain of the Citizen (Marglin 1963, p98).[9] One good reason for this is that individuals cannot provide personally for their distant descendants, because of the mixing of bequests that occurs over several generations. This point has been noted verbally by Daly and Cobb (1989, p39) and Howarth and Norgaard 1993, p351), and formal modelling might produce a firm basis for treating sustainability as a public good, which would largely resolve the conundrum.

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8. The apparent paradox is not as direct in an overlapping generations context. Society may have a view on distributing resources across generations to achieve an intergenerational sustainability objective, but this need not imply a constraint on any generation's maximisation of PV over its own, finite lifetime.

9. We would not go as far as Marglin by saying that "The Economic Man and the Citizen are for all intents and purposes two different individuals," since Economic Man can still maximize self-interest (seek individual optimality) within the bounds (sustainability) that the Citizen lays down. However, protests in 2000 over fuel price rises in many Western countries, despite those countries having recently signed the Kyoto Protocol which aims to limit greenhouse gas emissions, may perhaps be seen as a sign that the schizophrenia is real and can lead to quite disconnected behavior.

This paper's philosophical basis of neoclassical sustainability economics thus rejects classical utilitarianism, which prohibits any discounting; it rejects neoclassical utilitarianism, which sees maximising PV (with a constant ρ) as a complete prescription for intertemporal equity; and it also rejects purely rights-based view that it is future generations' resource opportunities, not utility outcomes that matter.

Some possible origins of this rather schizophrenic approach to sustainability can be seen in the long-running discussion in mainstream economics about calculating income. The most general economic concept of income is a measure of what the economy is now doing or enjoying that includes an adequate provision for the future, just as most individuals provide for their own future by investing some of their current money income rather than consuming it all. But asking how much is an adequate provision for the future immediately raises controversial, and arguably insoluble, questions about intergenerational equity. One can see a certain degree of schizophrenia in the classic discussion of income in Hicks (1946, Chapter 14). He sees income there as a guide to "prudent" behavior that will avoid "impoverishment" (and hence achieve sustainability, perhaps?), but also as something that, held constant, is equivalent to the present value of future receipts. This allows some writers to interpret 'Hicksian income' as some measure of sustainable income, and others to interpret it as wealth-equivalent income. These measures may well coincide at the level of a Hicksian individual, but they generally diverge at the macroeconomic level of most sustainability analyses.

6. CONCLUSIONS

By applying recent developments in the theory of national income and welfare measurement when there are multiple consumption goods to a representative agent, present-value-maximising economy, we have formally derived a pair of general, one-sided tests for the unsustainability of an economy. If augmented green net national product (GNNP) is momentarily constant or falling, or the value of augmented net investments is momentarily zero or negative, then at that moment the economy is enjoying a level of utility which cannot be sustained forever. (The 'green' in GNNP means that all environmental stocks and flows are included in the measure; while the 'augmented' means that the value of exogenous shifts of production possibilities that happen just by time passing, whether by technical progress or changing terms of trade, is also included.) There is no corresponding test for sustainability, and a well known 'folk' test, that of constant wealth, is shown in fact to imply unsustainability if the interest rate is constant.

Moreover, an economy with a single peak of development is very likely to go through a period when augmented net investment and GNNP change are strictly positive, but the economy is unsustainable. Existing models of national accounting with resource trade and exogenous technical progress were shown to be special cases of this paper's approach. A specific, example economy was used to show how to incorporate many issues in green accounting usually tackled separately, and incidentally showed the overlooked result that resource exploration and extraction costs must be deducted from GDP, as part of the adjustments needed to calculate GNNP. The net investment test probably requires less environmental valuation to be done than the GNNP change test, and then would be somewhat easier to use.

The practical difficulties of using the tests are no less than alternatives used in the past. There is no escape from the need to put dollar values on small changes in all useful environmental resources, no matter how disconnected these resources are from current markets. Indeed, the price of produced consumption can in general no longer be constant (and therefore set at unity), but will change over time in terms of a utility index of produced consumption and environmental goods, as was illustrated by an example with specific functional forms. The tests also complicate matters by reminding us of the need to account for future prospects for technical progress and changing terms of trade, as well as the environmental resource issues usually addressed by 'green' accounting. However, they should help avoid some of the more obvious theoretical shortcomings of national income accounting. The philosophical limitations of the tests are that it remains unexplained why sustainability should be of interest in a present-value-maximising economy, but this is a limitation shared by most previous literature. Treating sustainability formally as a public good is a possible future solution to this conundrum.

APPENDIX

Proof of Proposition 2

The proof varies slightly for the two conditions. Starting with the condition on the sign of net investment,

$$\mathbf{Q}^{\dagger}(t).\dot{\mathbf{K}}^{\dagger}(t) \leq 0 \;\Rightarrow\; \lambda(t)\pi(t)\mathbf{Q}^{\dagger}(t).\dot{\mathbf{K}}^{\dagger}(t) \leq 0$$

$$\Rightarrow\; \Psi^{\dagger}(t).\dot{\mathbf{K}}^{\dagger}(t) \leq 0 \text{ by [5]} \;\Rightarrow\; H(t) \leq U(t) \text{ by [4].} \qquad \text{[A1]}$$

Also $(d/ds)[H(s)e^{-\rho(s-t)}] = [\dot{H}(s) - \rho H(s)]e^{-\rho(s-t)} = \rho U(s)e^{-\rho(s-t)}$, where the second equality follows from [4]. Integrating this from t to ∞ gives

$$H(t) \;=\; \rho\int_t^\infty U(\mathbf{C}(s))e^{-\rho(s-t)}ds,$$

$$\Rightarrow\; H(t)/\rho \;=\; \int_t^\infty H(t)e^{-\rho(s-t)}ds \;=\; \int_t^\infty U(\mathbf{C}(s))e^{-\rho(s-t)}ds \;=\; W(t). \qquad [A2]$$

(The last two steps, and hence the proof, depend on the utility discount rate, ρ, being constant. If it is instead variable, $-\dot{\phi}(s)/\phi(s)$ replaces ρ as a multiplier, and the former cannot be taken outside the integral sign and transformed to a present value weighting for $H(t)$, as occurs here with ρ.) The non-constancy and uniqueness of the optimal path then means that

$$H(t) > U_m(t). \qquad [A3]$$

Otherwise, following the feasible constant utility path $U(s) = U_m(t)$ for all $s \geq t$ would, using the PV-equivalence result [A2], give at least the same PV as the (non-constant) optimal utility path, a contradiction of a unique optimum. Combining [A1] and [A3] gives the result that current $U(t)$ is unsustainable:

$$U(t) \geq H(t) > U_m(t) \qquad \text{as required in [14].}$$

Starting from the sign of GNNP change requires in addition Proposition 1 (after AW), and an assumption that the real interest rate $r(t)$ is positive, to be able to deduce the same sign for net investment:

$$\dot{Y}^\dagger(t) \leq 0 \;\Rightarrow\; r(t)\mathbf{Q}^\dagger(t).\dot{\mathbf{K}}^\dagger(t) \leq 0 \text{ by [11]} \;\Rightarrow\; \mathbf{Q}^\dagger(t).\dot{\mathbf{K}}^\dagger(t) \leq 0 \text{ by } r > 0.$$

The rest of the proof follows as above.

Proof of Proposition 3

$Y^\dagger(\mathbf{K}^\dagger,\mathbf{Q}^\dagger)$ is defined as max $\mathbf{P}.\mathbf{C} + \mathbf{Q}^\dagger.\dot{\mathbf{K}}^\dagger$ for $(\mathbf{C},\dot{\mathbf{K}}) \in \Pi(\mathbf{K}^\dagger)$.
Hence from Proposition 1,

$$r\mathbf{Q}^\dagger.\dot{\mathbf{K}}^\dagger = (d/dt)[Y^\dagger(\mathbf{K}^\dagger,\mathbf{Q}^\dagger)]$$
$$= (\partial Y^\dagger/\partial \mathbf{K}^\dagger).\dot{\mathbf{K}}^\dagger + (\partial Y^\dagger/\partial \mathbf{Q}^\dagger).\dot{\mathbf{Q}}^\dagger$$

$$= (\partial Y^\dagger/\partial \mathbf{K}^\dagger).\dot{\mathbf{K}}^\dagger + \dot{\mathbf{Q}}^\dagger.\dot{\mathbf{K}}^\dagger,$$

and since this is true for any general $\dot{\mathbf{K}}^\dagger$, for all components with $\dot{K}_i^\dagger \neq 0$,

$$rQ_i^\dagger - \dot{Q}_i^\dagger = \partial Y^\dagger/\partial K_i^\dagger. \qquad \text{[A4]}$$

Since $\dot{t} = 1$, $\dot{K}_i^\dagger \neq 0$ for the time component of [A4], which is

$$rQ^t - \dot{Q}^t = \partial Y^\dagger/\partial t = \partial Y/\partial t = Y_t \qquad \text{[A5]}$$

because $Q^t = Y^\dagger - Y$ has no exogenous time dependence. [A5] can be integrated from time t to ∞ to give

$$Q^t(t) = \int_t^\infty [\partial Y(s)/\partial s]\exp[-\int_t^s r(z)dz]ds \qquad \text{which is [16].}$$

Proof of Proposition 8

The current value Hamiltonian of the dynamic optimisation problem of maximising wealth[10] is

$$Y^\dagger(t) := Y(t) + Q^t = P^C C + P^B B + \mathbf{Q}^\dagger.\dot{\mathbf{K}}^\dagger \qquad \text{[A6]}$$

where

$\mathbf{K}^\dagger := (K, K_a, K_f, \Omega, \boldsymbol{S}, t)$ is the vector of all state variables; [A7]

$\mathbf{Q}^\dagger := (Q^K, Q^a, Q^f, Q^\Omega, Q^{\boldsymbol{S}}, Q^t)$ is the vector of corresponding co-state variables (shadow consumption prices of stocks).

The prices and investment flows defined by [26]-[34] then make

$$\begin{aligned} Y^\dagger(t) &= P^C C + P^B B + Q^K\dot{K} + Q^a\dot{K}_a + Q^f\dot{K}_f + Q^\Omega\dot{\Omega} + Q^{\boldsymbol{S}}.\dot{\boldsymbol{S}} + Q^t && \text{[A8]} \\ &= P^C C + P^B\{B_0 - \varepsilon^B E^B[F(K,\boldsymbol{R}\text{-}\boldsymbol{R}_x,t),a,K_a,t] - \varepsilon^\Omega(\Omega_1 - \Omega)\} \\ &\quad + Q^K[F(K,\boldsymbol{R}\text{-}\boldsymbol{R}_x,t) + M - C - \delta K - a - I_a - V(\boldsymbol{D},\boldsymbol{S},t) - X(\boldsymbol{R},\boldsymbol{S},t)] \\ &\quad + Q^a[I_a - \delta K_a] + Q^f[r(t)K_f + \boldsymbol{Q}^x(t).\boldsymbol{R}_x - M] \\ &\quad + Q^\Omega[\gamma(\Omega) - E^\Omega(\boldsymbol{R}\text{-}\boldsymbol{R}_x)] + Q^{\boldsymbol{S}}.[\boldsymbol{D} + \boldsymbol{G}(\boldsymbol{S}) - \boldsymbol{R}] + Q^t, && \text{[A9]} \end{aligned}$$

so the first order conditions with respect to a set of independent control

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10. We assume that the optimal (welfare-maximising) path is regular, in the sense of Asheim (1997, p368), so that it maximizes wealth as well as welfare.

variables (I is left as the dependent variable, given by [29]) are

$\partial Y^{\dagger}/\partial C = P^C - Q^K = 0 \quad \Rightarrow \quad Q^K = P^C$ [A10]

$\partial Y^{\dagger}/\partial a = P^B B_a - Q^K = 0 \quad \Rightarrow \quad P^B/P^C = 1/B_a = b$ [A11]

$\partial Y^{\dagger}/\partial I_a = -Q^K + Q^a = 0 \quad \Rightarrow \quad Q^a = Q^K = P^C$

$\partial Y^{\dagger}/\partial \boldsymbol{D} = -Q^K V_{\boldsymbol{D}} + Q^{\boldsymbol{S}} = \boldsymbol{0} \quad \Rightarrow \quad Q^{\boldsymbol{S}}/Q^K = V_{\boldsymbol{D}}$ [A12]

$\partial Y^{\dagger}/\partial \boldsymbol{R} = P^B B_{\boldsymbol{R}} + Q^K(F_{\boldsymbol{R}} - X_{\boldsymbol{R}}) - Q^{\Omega} E^{\Omega}{}_{\boldsymbol{R}} - Q^{\boldsymbol{S}} = \boldsymbol{0}$

$\Rightarrow \; Q^{\Omega} E^{\Omega}{}_{\boldsymbol{R}}/Q^K = (P^B/P^C) B_{\boldsymbol{R}} + F_{\boldsymbol{R}} - X_{\boldsymbol{R}} - Q^{\boldsymbol{S}}/Q^K$

$\Rightarrow \; Q^{\Omega}/Q^K = (bB_{\boldsymbol{R}} + F_{\boldsymbol{R}} - X_{\boldsymbol{R}} - V_{\boldsymbol{D}})_i \,/\, (E^{\Omega}{}_{\boldsymbol{R}})_i$, for all i [A13]

$\partial Y^{\dagger}/\partial M = Q^K - Q^f = 0 \quad \Rightarrow \quad Q^f = Q^K = P^C$ [A14]

$\partial Y^{\dagger}/\partial \boldsymbol{R}_x = -P^B B_{\boldsymbol{R}} - Q^K F_{\boldsymbol{R}} + Q^f \boldsymbol{Q}^x + Q^{\Omega} E^{\Omega}{}_{\boldsymbol{R}} = 0$; then use [A11], [A14], [A12]:

$\Rightarrow \; P^B B_{\boldsymbol{R}} + Q^K F_{\boldsymbol{R}} - Q^{\Omega} E^{\Omega}{}_{\boldsymbol{R}} = P^C \boldsymbol{Q}^x = Q^K X_{\boldsymbol{R}} + Q^{\boldsymbol{S}}$ [A15]

$\Rightarrow \; bB_{\boldsymbol{R}} + F_{\boldsymbol{R}} - (Q^{\Omega}/Q^K) E^{\Omega}{}_{\boldsymbol{R}} = \boldsymbol{Q}^x = X_{\boldsymbol{R}} + V_{\boldsymbol{D}}$ [A16]

For Q^t, first use [A6] and [A9] to get

$\partial Y/\partial t = P^B B_t + Q^K(F_t - V_t - X_t) + Q^f(\dot{r} K_f + \dot{\boldsymbol{Q}}^x . \boldsymbol{R}_x)$

which, after using [A10], [A11] and [A14] becomes

$\partial Y/\partial t = P^C(bB_t + F_t - V_t - X_t + \dot{r} K_f + \dot{\boldsymbol{Q}}^x . \boldsymbol{R}_x)$

hence from [16],

$Q^t(t) := \int_t^{\infty} P^C(s) \,\{bB_t + F_t - V_t - X_t + \dot{r} K_f + \dot{\boldsymbol{Q}}^x . \boldsymbol{R}_x\}(s) \exp[-\int_t^s r(z) dz] \, ds$

which is [36].

Inserting [A10]-[A16] into a cross between [A8] and [A9] then gives

$Y^{\dagger} = P^C C + P^C bB + Q^K\{\dot{K} + \dot{K}_a + \dot{K}_f\} + Q^{\Omega}\dot{\Omega} + Q^{\boldsymbol{S}} . \dot{\boldsymbol{S}} + Q^t$

which using [16], [A12] and [A16] gives

$= P^C \{C + bB + \dot{K} + \dot{K}_a + \dot{K}_f + (bB_R + F_R - \boldsymbol{Q}^x)_i \dot{\Omega}/(E^{\Omega}{}_{\boldsymbol{R}})_i$

$+ (\boldsymbol{Q}^x - X_{\boldsymbol{R}}) . \dot{\boldsymbol{S}} \} + Q^t$ which is [35].

If the problem is autonomous, time is 'unproductive,' so its value Q^t, the last term of [35], disappears.

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REFEREES' APPENDIX

(1) PROOF OF PROPOSITION 7 (single-peakedness)

Here we use asterisks (*) to denote all optimal values, to distinguish them from hypothetical, non-optimal ones. Let the single peak of optimal utility be at time $T_P > 0$; so $\dot{U}^*(t) \gtrless 0$ for $0 \le t \lessgtr T_P$. Three building blocks of the proof are first, $t \ge T_P \Rightarrow W^*(t) = \int_t^\infty U(\mathbf{C}^*(s))e^{-\rho(s-t)}ds < \int_t^\infty U(\mathbf{C}^*(t))e^{-\rho(s-t)}ds = U(\mathbf{C}^*(t))/\rho$. Using the present value equivalence of $H^*(t)$ proved in [A2], and the fact that the sustainable utility path cannot exceed the optimal path in present value, this means

$$t \ge T_P \;\Rightarrow\; U_m(t) < H^*(t) < U^*(t) \qquad \text{[A17]}$$

Second, whenever the economy is unsustainable, sustainable utility cannot be rising:

$$U_m(t) < U^*(t) \;\Rightarrow\; \dot{U}_m(t) \le 0 \qquad \text{[A18]}$$

For if $\dot{U}_m > 0$, then $U_m(t+\varepsilon_1) > U_m(t)$ for some $\varepsilon_1 > 0$. Hence one could strictly dominate the constant utility path $U(s) = U_m(t)$, all $s \ge t$, by following $U(s) = U^*(s)$, $t \le s < t+\varepsilon_1$, and then $U(s) = U_m(t+\varepsilon_1)$, $s \ge t+\varepsilon_1$. But this would contradict the assumed intertemporal Pareto efficiency of $U(s) = U_m(t)$, all $s \ge t$.

Third, if an optimal path is unsustainable at s where $T_P - \varepsilon_2 < s < T_P$, for some $\varepsilon_2 > 0$, it remains unsustainable between s and T_P, by [A18] and because $\dot{U}^*(t) > 0$ during this period. Together with [A17], this means that once an optimal, single-peaked path becomes unsustainable, it stays unsustainable.

Hence, if the optimal path is initially unsustainable (*Case 1*), it is unsustainable for all t. If however the path is initially sustainable (*Case 2*), then by a continuity argument and [A17] there is a last time T_L with $0 \le T_L < T_P$ when it is still sustainable. That is, $U_m(t) \gtrless U^*(t)$ for $t \lessgtr T_L$.

The given assumption that net investment is initially positive means by [4] that $H^*(0) > U^*(0)$. But $H^*(T_P) < U^*(T_P)$ by [A17]. By a continuity argument, there is then a time T_H, $0 < T_H < T_P$, for which $t \lessgtr T_H \;\Rightarrow\; H^*(t) \gtrless U^*(t)$. In Case 1, $0 < t < T_H$ is then the finite period during which net investment is positive but utility is unsustainable. In Case 2, we must have $T_L < T_H$, since $T_L \ge T_H$ would imply the existence of a time s, $T_H \le s \le T_L$ when $U^*(s) \le U_m(s)$ (sustainability), $U_m(s) < H^*(s)$ (present-value-equivalence and uniqueness), and yet $H^*(s) \le U^*(s)$, a contradiction. $T_L < t < T_H$ is then a finite period of positive net investment yet unsustainability.

(2) PROOF OF RESULTS IN SECTION 4

The current value Hamiltonian of the dynamic optimisation problem of choosing C, a and R over time to maximise wealth is

$$Y^\dagger(t) := Y(t) + Q^t = P^C C + P^B B + Q^K \dot{K} + Q^S \dot{S} + Q^t \dot{t}$$
$$= P^C C + P^B a/R + Q^K(K^\alpha R^\beta e^{\nu t} - C - a) - Q^S R + Q^t \qquad \text{[R1]}$$

so the first order conditions with respect to C and a are

$\partial Y^\dagger/\partial C = P^C - Q^K = 0 \quad \Rightarrow \quad Q^K = P^C$

$\partial Y^\dagger/\partial a = P^B B_a - Q^K = 0 \quad \Rightarrow \quad P^B/P^C = 1/B_a = b = R$ [R2]

[5] $\quad \Rightarrow \quad P^C/P^B = U_C/U_B$, and hence

[R2],[42] $\quad \Rightarrow \quad 1/b = \alpha B/\varepsilon C \;\Rightarrow\; bB = a = (\varepsilon/\alpha)C$ [47]

whilst the other three first order conditions are

$\partial Y^\dagger/\partial R = P^B B_R + Q^K \beta F/R - Q^S = 0 \quad \Rightarrow \quad Q^S = RP^C(-a/R^2) + P^C \beta F/R$

$= P^C[\beta F - (\varepsilon/\alpha)C]/R.$ [54]

$\partial Y^\dagger/\partial K = Q^K \alpha F/K = rQ^K - \dot{Q}^K \quad \Rightarrow \quad \dot{P}^C/P^C = r - \alpha F/K$ [R3]

$\partial Y^{\dagger}/\partial S = 0 = rQ^S - \dot{Q}^S \quad \Rightarrow \quad \dot{Q}_S/Q_S = r = \dot{P}^C/P^C + \dot{F}/F - \dot{R}/R$ [R4]

where we have assumed a balanced growth path with $\dot{a}/a = \dot{F}/F$.

To find the exponents on this path, try

$C = C_0e^{\omega t}$, $K = K_0e^{\omega t}$, $a = a_0e^{\omega t}$, $S = S_0e^{-\zeta t}$, $R = \zeta S_0e^{-\zeta t}$, $P^C = P_0e^{\eta t}$ [R5]

[R4],[R5] $\Rightarrow$ $r = \omega + \zeta + \eta$, while [R6]

[5],[8] $\Rightarrow$ $r = \rho - \dot{U}_C/U_C + \dot{P}^C/P^C$, which using [42] and [R5]

$= \rho + \dot{C}/C + \eta = \rho + \omega + \eta.$ [R7]

[R6],[R7] $\Rightarrow$ $\zeta = \rho$, $r = \rho + \omega + \eta$ [50]

Equating powers of e^t and levels in [R3] then gives

$(\alpha-1)\omega - \beta\rho + \nu = 0$ and

$\alpha K_0^{\alpha-1}S_0^{\beta} = \rho + \omega + \eta - \eta = \rho + \omega$ [R9]

$\Rightarrow$ $(\rho+\omega)K_0^{1-\alpha} = \alpha S_0^{\beta}$ where $\omega = (\nu-\beta\rho)/(1-\alpha)$, [43]

From the production function, [R9], [R5] and [47],

$F = (\alpha F/K)K/\alpha = (\rho+\omega)K/\alpha = \omega K + C + (\varepsilon/\alpha)C$

$\Rightarrow$ $C = \alpha a/\varepsilon = \{[\rho+(1-\alpha)\omega]/(\alpha+\varepsilon)\}K.$ [44]

and hence $F = \{\omega(\alpha+\varepsilon)/[\rho+(1-\alpha)\omega] + (\alpha+\varepsilon)/\alpha\}C$

$= (\alpha+\varepsilon)\{\alpha\omega + [\rho+(1-\alpha)\omega]\}C/\alpha[\rho+(1-\alpha)\omega]$

$= (\alpha+\varepsilon)(\rho+\omega)C/\alpha[\rho+(1-\alpha)\omega]$ [R10]

Also

[R1] $\Rightarrow$ $\partial Y/\partial s = \partial(Y^{\dagger}-Q^t)/\partial s = P^C(\partial F/\partial s) = P^C\nu F$

$\Rightarrow$ $(\partial Y/\partial s)e^{-r(s-t)} = P_0e^{\eta s}\nu F_0e^{\eta s}e^{-r(s-t)} = \nu P_0F_0e^{rt}e^{-(r-\eta-\omega)s}$

[50],[16] $\Rightarrow$ $Q^t = \nu P_0F_0e^{rt}\int_t^{\infty}e^{-\rho s}ds = (\nu P_0F_0/\rho)e^{(\eta+\omega)t} = \nu P^CF/\rho$ [R11]

[R10],[R11] $\Rightarrow$ $Q^t = \{\nu(\alpha+\varepsilon)(\rho+\omega) / \rho\alpha[\rho+(1-\alpha)\omega]\}\ P^CC.$ [51]

[19],[R2] $\Rightarrow$ $Y_e(t) = \int_t^{\infty}P^C(s)[C(s)+b(s)B(s)]e^{-rs}ds \,/\, \int_t^{\infty}e^{-rs}ds$

[47],[R5] $\Rightarrow$ $Y_e(t) = P_0(1+\varepsilon/\alpha)C_0\int_t^{\infty}e^{-(r-\eta-\omega)s}ds \,/\, (e^{-rt}/r)$

[50] $\Rightarrow$ $Y_e(t) = [(\rho+\omega+\eta)P_0(1+\varepsilon/\alpha)C_0/\rho]e^{(r-r+\eta+\omega)t}$

[R5] $\Rightarrow$ $Y_e(t) = [(\rho+\omega+\eta)(1+\varepsilon/\alpha)/\rho]P^C C$

$= [\eta(1+\varepsilon/\alpha)/\rho + (\rho+\omega)(\alpha+\varepsilon)/\alpha\rho]P^C C$

[48] $\Rightarrow$ $Y_e(t) = [\varepsilon/\alpha + (\alpha+\varepsilon)(\rho+\omega)/\alpha\rho]\ P^C C$ [52]

[51],[52] $\Rightarrow$ $Y = Y_e - Q^t$

$= \{\varepsilon/\alpha + (\alpha+\varepsilon)(\rho+\omega)/\alpha\rho - \nu(\alpha+\varepsilon)(\rho+\omega) / \rho\alpha[\rho+(1-\alpha)\omega]\}\ P^C C$

$= \{\varepsilon/\alpha + (\alpha+\varepsilon)\mathbf{[}\rho+\omega - \nu(\rho+\omega) / [\rho+(1-\alpha)\omega]\mathbf{]}/\alpha\rho\}\ P^C C$

$= \{\varepsilon/\alpha + (\alpha+\varepsilon)(\rho+\omega)\mathbf{[}1 - \nu/[\rho+(1-\alpha)\omega]\mathbf{]}/\alpha\rho\}\ P^C C$

$= \{\varepsilon/\alpha + (\alpha+\varepsilon)(\rho+\omega)\mathbf{[}\rho+(1-\alpha)\omega - \nu\mathbf{]}/\alpha\rho[\rho+(1-\alpha)\omega]\}\ P^C C$

use [43] $= \{\varepsilon/\alpha + (\alpha+\varepsilon)(\rho+\omega)\mathbf{[}\rho+\nu-\beta\rho-\nu\mathbf{]}/\alpha\rho[\rho+(1-\alpha)\omega]\}\ P^C C$

$= \{\varepsilon/\alpha + (\alpha+\varepsilon)(\rho+\omega)(1-\beta)/\alpha[\rho+(1-\alpha)\omega]\}\ P^C C.$ [53]

The same result can be reached from $Y = \{C+bB+\dot{K}-R[\beta F-(\varepsilon/\alpha)C]/R\}P^C$ by using [47], [R5], [44], [R10] and [54].

[42],[R5],[47] $\Rightarrow \dot{U}$ $= (\alpha/C)\dot{C} + (\varepsilon/B)\dot{B} = \alpha\omega + \varepsilon(\dot{a}/a-\dot{R}/R)$

$= \alpha\omega + \varepsilon(\omega+\rho) = (\alpha+\varepsilon)\rho\ [\omega/\rho + \varepsilon/(\alpha+\varepsilon)]$ [55]

[R5],[54],[R11] $\Rightarrow Q^K\dot{K} - Q^S R + Q^t = \{\omega K - \beta F + \varepsilon C/\alpha + \nu F/\rho\}P^C$

use [R10] $= \{\omega\alpha/(\rho+\omega) - \beta + [\rho+(1-\alpha)\omega]\varepsilon/(\alpha+\varepsilon)(\rho+\omega) + \nu/\rho\}P^C F$

use [43] $= \{\omega\alpha/(\rho+\omega) + (1-\alpha)\omega/\rho + [\rho+(1-\alpha)\omega]\varepsilon/(\alpha+\varepsilon)(\rho+\omega)\}P^C F$

$= \{[\rho\alpha+(1-\alpha)(\rho+\omega)]\omega/\rho + [\rho+(1-\alpha)\omega]\varepsilon/(\alpha+\varepsilon)\}P^C F/(\rho+\omega)$

$= \{\omega/\rho + \varepsilon/(\alpha+\varepsilon)\}P^C F[\rho+(1-\alpha)\omega]/(\rho+\omega)$

use [R10] $= (1+\varepsilon/\alpha)\ [\omega/\rho + \varepsilon/(\alpha+\varepsilon)]P^C C.$ [56]

[ends]