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Abstract. Exact optimal paths are calculated for a closed economy with human-made capital, non-renewable resource depletion and exogenous technical progress in production, hyperbolic utility discounting, and (possibly) hyperbolic technical progress. On its optimal path, generally, welfare-equivalent income > wealth-equivalent income > Sefton-Weale income > NNP, with possibly dramatic differences among these measures; and sustainable income can be greater, equal or less than NNP. This supports the view that there can be no best, exact definition of income. For low enough discounting, growth is optimal even when technical progress is zero. A particular discount rate makes all income measures and consumption constant and (except NNP) equal; and zero technical progress then gives the Solow (1974) maximin as a special case. General problems with calculating sustainable income when there is technical progress are discussed, and the optimal path is time-consistent if the discount rate can depend on the economy's stocks and absolute time.

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Exact measures of income in a hyperbolic economy

JOHN C. V. PEZZEY

Summary

This paper illustrates both how different five measures of income can be, and how important hyperbolic discounting can be in avoiding a dilemma between two classic approaches to intergenerational equity. The illustration uses a theoretical, non-trading economy with an explicit functional form for the dependence of production on a stock of human-made capital, a flow of non-renewable resource depletion, and time in the form of an exogenous technical progress factor. Results are calculated for the economy's 'optimal' path that maximises the sum of the discounted wellbeing (utility) of a typical person over the rest of time. The discount is hyperbolic, meaning it declines as the inverse of a linear function of time, rather than being a constant as usually assumed. The rate of technical progress in production is also hyperbolic, with the same decline over time as the discount rate.

The five measures of income considered are welfare-equivalent income, wealth-equivalent income, sustainable income and net national product (NNP), all as reviewed by Asheim (2000); and Sefton-Weale income, after Sefton and Weale (1996). On the optimal path, welfare-equivalent income, wealth-equivalent income, Sefton-Weale income and NNP at any moment form a strictly decreasing series of values, except in the special case where optimal consumption is constant. With no technical progress, one can also show that NNP and sustainable income are not generally equal. A plausible numerical example reveals dramatic differences among the measures, with for example wealth-equivalent income being initially about 15 times sustainable income, and forever about 20 times NNP. These clear differences between income measures are seen as support for the view that

there can never be a best, exact definition of income commanding universal assent, because there are many different purposes in measuring income.

The two classic approaches to intergenerational equity in an economy with capital and non-renewable resources are maximin, which yields constant consumption, and optimality using a constant utility discount rate. The dilemma in choosing between them in the case of no technical progress is that constant consumption (and hence wellbeing) prevents any growth, whereas constant discounting leads to a long run decline in consumption, and hence wellbeing. For a low enough discount rate, the hyperbolic economy avoids this dilemma by allowing sustained growth of consumption. The Solow (1974) constant consumption solution is in fact a special case of the hyperbolic economy, with zero technical progress and a discount rate just high enough to prevent growth. Some notes are also given on how to calculate sustainable income numerically when there is positive technical progress.

Unlike for constant discounting, there is no axiomatic foundation available to justify why an economy would be motivated to follow a path with hyperbolic discounting. However, the resulting optimal path is shown to be time-consistent, provided one breaks the convention that the discount factor should depend only on relative time and psychological parameters. For time-consistency under hyperbolic discounting, it it necessary that the discount factor varies with the economy's productive stocks and with absolute time.

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1. Introduction

This paper gives exact formulae for four or five different definitions of income on the optimal development path in a theoretical economy with explicit functional forms. The economy is closed and deterministic, with constant population and a representative agent, and the optimal path is that which maximises the present value of utility over an infinite time horizon. There are three inputs to production: the stock of human-made capital, the depletion of a finite, non-renewable resource, and time in the form of (exogenous) technical progress. The utility discount factor and technical progress factor are both hyperbolic rather than exponential functions of time, so it will be called 'the hyperbolic economy' below.

Because of this economy's explicit functional forms, the purposes of studying it need careful explanation. One is to show the ethically attractive property that as long as discounting is hyperbolic and small enough, forever rising rather than constant consumption can be the optimal (present-value-maximising) development path of an economy with human-made capital and a non-renewable resource, even with no technical progress. This shows how hyperbolic discounting can solve the well-known problem, that a maximum constant consumption path may perpetuate poverty or be foolishly conservative (Solow 1974), without causing the intergenerational equity that may result from constant discounting, for example in Dasgupta and Heal (1974).

Another, perhaps more important, purpose is to give a clear example, inspired by the general theory reviewed by Asheim (2000), of why there can be no single, exact definition of income. This will come from being able to show analytically, that four out of five income measures have strictly different sizes in the hyperbolic economy; and numerically, that the size differences can be dramatic for plausible parameter values. The hyperbolic economy may also prove a useful testbed for the recent interest in hyperbolic and other non-constant discounting, especially for the far-distant future (see for example Henderson and Bateman 1995, Laibson 1997 and Weitzman 2001). It yields some additional insights into the estimation of sustainable income when there is technical progress, and the definition of time consistency. Finally, the hyperbolic economy adds to the range of algebraically exact economies which can be used to develop or check new theories about economies with both capital and non-renewable resources, and perhaps to reveal the often limited generality of existing theories. This range otherwise seems to comprise only Solow's constant consumption solution, the asymptotic steady state in Stiglitz (1974), and Pezzey and Withagen's (1998) solution of a 'single-peaked' economy.

As a preliminary to illustrate the problem of limited generality, Section 2 lists ten features of an economy with human-made capital and natural resources, which are hardly ever all fully general in well-known theoretical results in the literature on optimal development and income measurement. Section 3 defines the hyperbolic economy, lists and interprets its results, and discusses whether or not its optimal path is well-motivated and time-consistent. All calculations use straightforward, though tedious and hence omitted, algebra (flagged by "it can be shown that..."), that starts from the necessary first order conditions of the optimal control problem; full details are available from the author. Section 4 concludes.

2. Ten sources of non-generality in theoretical results

Any new features in the hyperbolic economy spring from the lack of full generality found in almost all theoretical models of economies with both human-made capital and natural resources, even when these are confined to representative-agent models where population is constant and consumption is the sole determinant of utility. For example, two of the best known results of the mid-1970s use significantly different assumptions, which conceals their interrelationship within the more general theory summarised by Asheim (2000). Weitzman's (1976) result, on the annuity-equivalent properties of net national product, assumes non-linear production, non-constant consumption, a linear utility function and a constant interest rate. Hartwick's (1977) rule, on constant consumption forever resulting from zero net investment forever, assumes linear production, constant consumption, and (implicitly) a non-linear utility function and a declining interest rate. The hyperbolic economy here makes the same assumptions as Hartwick, except that consumption can be constant, rising or falling.

As a reminder of these kinds of differences, Table 1 lists ten key features about production functions, utility functions, intertemporal objectives and trade, and simplifying assumptions which are often made about them. The notation used is fairly standard, and is fully defined in the next section.

Table 1 Ten key features, some of which are simplified in almost all theoretical models of economies with human-made capital and natural resources

No.	General feature	Simplifying assumption
1	Non-linear consumption/ investment frontier	Linear consumption/investment frontier (e.g. $F = C + \dot{K}$)
2	Resource discovery and extraction costs	No resource discovery or extraction costs
3	Capital depreciation	No capital depreciation
4	Unspecified returns to scale in production	Constant returns to scale in production
5	Exogenous technical progress	No exogenous technical progress
6	Non-linear utility function	Linear utility function (i.e. $U = C$)
7	Non-constant utility discount rate	Constant utility discount rate $\rho > 0$ (i.e. discount factor $\phi(t) = e^{-\rho t}$)
8	Non-constant interest rate	Constant interest rate r
9	Closed or large open economy (so prices are endogenous)	Small open economy (so prices are exogenous)
10	No constant consumption goal	Constant consumption goal, $\dot{C} = 0$

3. The hyperbolic economy

3.1 General assumptions and definitions of income

The economy is a special case of that described in the appendix of Asheim (1997). Population is constant; consumers are identical and have no age structure, with each generation represented by one agent at an instant in continuous time, which stretches from zero to infinity; and the economy is closed to trade. The variables below are non-negative quantities along any development path in the economy, using terminology similar to that in

Asheim (2000). Less familiar terms, or ones which are often given different meanings in the literature, are highlighted in italics.

$K(t)$ is the non-depreciating, human-made capital stock; $K(0) = K_0 > 0$

$S(t)$ is the non-renewable, natural resource stock; $S(0) = S_0 > 0$

$C(t)$ is consumption of a single produced good

$R(t) = -\dot{S}(t)$ is the resource depletion flow, with zero extraction costs

$F(K(t),R(t),t)$ is output; $F = F(K(t),R(t))$ if technology is constant

$U(C(t))$ is instantaneous utility

$\phi(t)$ is the utility discount factor

$\Phi(t) := \phi(t)U_C(C)$ is the consumption discount factor

$W(t) := \int_t^\infty [\phi(s)/\phi(t)]U[C(s)]ds,\ t \geq 0$ is (current) *welfare*. The representative agent chooses consumption and resource depletion paths to maximise welfare $W(0)$, and the resulting path is called optimal. Existence and uniqueness are assumed.

$\mu^K(t)$, $\mu^S(t)$ are respectively the co-state variables of $K(t)$ and $S(t)$ resulting from this optimisation.

$\Theta(t) := \int_t^\infty [\Phi(s)/\Phi(t)]C(s)ds,\ t \geq 0$ is (current) *wealth*

$\delta(t) := -\dot{\phi}(t)/\phi(t)$ is the current (utility) discount rate

$\delta_\infty(t) := \int_t^\infty \phi(s)\delta(s)ds \,/\, \int_t^\infty \phi(s)ds$ is the *time-averaged discount rate*

$r(t) := -\dot{\Phi}(t)/\Phi(t)$ is the current interest rate

$r_\infty(t) := \int_t^\infty \Phi(s)r(s)ds \,/\, \int_t^\infty \Phi(s)ds$ is the *time-averaged interest rate*.

Five definitions of income are then

$A(t)$ $:= U^{-1}[\delta_\infty(t)W(t)]$ is *welfare-equivalent income* (Asheim 2000)

$Y_e(t)$ $:= r_\infty(t)\Theta(t)$ is *wealth-equivalent income* (Asheim 2000)

$SW(t)$ $:= [\int_t^\infty r(s)\Phi(s)C(s)ds] \,/\, \Phi(t)$ is *Sefton-Weale income*, after Sefton and Weale (1996)

$Y(t)$	$:= C(t) + [\mu^K(t)\dot{K}(t) + \mu^S(t)\dot{S}(t)]/U_C(t)$ is net national product (NNP)
$Y_m[K(t),S(t)]$	$:=$ max C s.t. $C(t') \geq C$ for all $t' \geq t$, i.e. *sustainable income* or the maximum sustainable consumption level. Y_m is calculated only when there is no technical progress, because an analytic solution is generally unavailable when there is technical progress (a matter we defer discussing until later).

An immediate question is whether one can decide which, if any, of these definitions gives the 'best' measure of income that we 'should' use. As already suggested, the view here is that there is no best measure; but we also defer discussing this until values of the five income measures for the hyperbolic economy, both analytically and for a numerical example, have been derived.

3.2 Specific assumptions and the optimal path for the hyperbolic economy

The specific functional forms used in the hyperbolic economy are:

Production:	$F = K^\alpha R^\beta (1+\theta t)^\nu = \dot{K}+C,\ \theta > 0,\ \nu \geq 0$	)	
Instantaneous utility:	$U(C) = C^{1-\alpha}/(1-\alpha), \quad 0 < \alpha < 1$	)	[1]
Discount factor:	$\phi(t) = (1+\theta t)^{-\rho}, \ \rho > 0$	)	

The hyperbolic utility discount factor $(1+\theta t)^{-\rho}$ a third question that we will discuss later: what would motivate the economy to maximise welfare $W(.)$ as defined above, given that non-exponential discounting is 'known' to make the welfare-maximising path time-inconsistent (Strotz 1955/6)? This is a third topic deferred for discussion later. The hyperbolic factor for exogenous technical progress (the $(1+\theta t)^{\nu}$ term in $F(.)$) is necessary to reach an exact solution, given the discount factor $(1+\theta t)^{-\rho}$. However, since the progress rate $\nu\theta/(1+\theta t)$ is positive but declining over time, it can also be viewed as a compromise between the usual assumptions of zero progress, or a constant, positive rate of progress.

Further necessary parameter restrictions, and algebraic abbreviations, are:

$\beta < \alpha < \alpha+\beta \leq 1$ ($\beta < \alpha$ is needed to enable a constant consumption path in Solow (1974)) [2]

$\rho > 1+\alpha-\beta+\nu$ (> 1) (needed for convergence of welfare W) [3]

$\xi := (\rho-\alpha-\nu)/(1-\beta)$ (> 1) [4]

$\sigma := (\alpha+\nu-\beta\rho)/(1-\alpha)(1-\beta)$ [5]

$\Rightarrow \xi+\sigma = \rho+\alpha\sigma = [\rho(1-\alpha-\beta)+\alpha(\alpha+\nu)] / (1-\alpha)(1-\beta)$ (> 0)

$\theta := [\alpha(\xi-1)^{\beta} S_0^{\beta}/(\xi+\sigma)K_0^{1-\alpha}]^{1/(1-\beta)}$ (> 0) [6]

Definition [6], which relates θ not just to the functional parameters α, β, ξ and σ, but also to the initial stock parameters K_0 and S_0, is very restrictive. It is needed to place the economy exactly on a (hyperbolically) steady state path from time zero. Without it, only steady state rates of growth can be computed analytically, much as in Stiglitz (1974).

It can be shown that the optimal (welfare-maximising) paths are then:

Consumption	$C(t)$	$= [(\rho-\alpha)\theta K_0/\alpha]\ (1+\theta t)^{\sigma}$	[7]
Capital	$K(t)$	$= K_0(1+\theta t)^{\sigma+1}$	[8]
Resource stock	$S(t)$	$= S_0(1+\theta t)^{-(\xi-1)}$	[9]
Resource flow	$R(t)$	$= (\xi-1)\theta S_0(1+\theta t)^{-\xi}$	
Output	$F(t)$	$= [(\xi+\sigma)/(\rho-\alpha)]C(t)$	
Current interest rate	$r(t)$	$= (\xi+\sigma)\theta/(1+\theta t)$	[10]
Time-averaged interest rate	$r_{\infty}(t)$	$= (\xi+\sigma-1)\theta/(1+\theta t)$	

3.3 The five measures of income for the hyperbolic economy

From the above results, it can further be shown that the five measures of income on the optimal path of the hyperbolic economy are at any time:

For any rate of technical progress, $\nu \geq 0$:

Welfare-equivalent income	$A(t)$	$= [1+(1-\alpha)\sigma/(\xi-1)]^{1/(1-\alpha)}C(t)$	[11]
Wealth-equivalent income	$Y_e(t)$	$= [1+\sigma/(\xi-1)]\ C(t)$	[12]
Sefton-Weale income	$SW(t)$	$= (1+\sigma/\xi)\ C(t)$	[13]
NNP	$Y(t)$	$= [1-\nu/(\rho-\alpha)](1+\sigma/\xi)\ C(t)$	[14]

For $\alpha > \beta$, *and no technical progress,* $\nu = 0$, *only:*

Sustainable income $Y_m(t) = [(\xi+\sigma)(\alpha-\beta)/(\xi-1)\alpha]^{\beta/(1-\beta)}(1+\sigma/\xi)C(t)$ [15]

Four algebraic features of these results are worth noting:

(a) Since all parameters are positive, as are $(1-\alpha)$, $(\xi-1)$ and $(\rho-\alpha)$ thanks to [2]-[4], the first four income measures are in the strict size order $A > Y_e > SW > Y$, consistent with the non-strict general order given in Asheim (2000). Finding general conditions for this strict order to hold remains for further work.

(b) The $-\nu/(\rho-\alpha)$ term in NNP, and its absence in welfare-equivalent, wealth-equivalent and Sefton-Weale incomes, clearly reflects the 'technical progress premium', which is overlooked by the national accounting definition of income but included in present-value-equivalent definitions. However, it remains to be seen if Weitzman's (1997) formula for the technical progress premium, which holds for an economy with a constant interest rate, can be generalised to the non-constant interest rate here.

(c) It can be shown that if $\alpha > \beta$, and there is no technical progress ($\nu = 0$):

$$\alpha/\beta \gtrless \rho\ (> 1) \iff Y_m \gtrless Y, \qquad [16]$$

so that sustainable income Y_m is only loosely related to NNP Y.

(d) From [5], if $\alpha+\nu-\beta\rho = 0$, then $\sigma = 0$ and all measures of income are constant, at levels which can be shown to be:

$$A(t) = Y_e(t) = SW(t) = Y_m(t) = C(t) = Y(t)/[1-\nu/(\rho-\alpha)] = \bar{C} \quad \forall t \geq 0$$

where

$$\bar{C}(\alpha,\beta,\nu,K_0,S_0) := [\alpha(1-\beta)+\nu]\{K_0^{\alpha-\beta}[\alpha(\alpha+\nu-\beta)S_0]^{\beta}/(\alpha+\nu)\}^{1/(1-\beta)} \quad [17]$$

Then if $\nu > 0$ (technical progress), the economy can forever consume (C) more than it 'produces' (Y), because time is itself productive but the value of time (i.e. of technical progress) is omitted from Y. ($\bar{C}$ then appears to be the first known algebraic expression for sustainable income Y_m in the case of technical progress, albeit restricted to the special case where $\theta = \alpha\beta\{[\alpha(\alpha+\nu-\beta)S_0]^{\beta}/(\alpha+\nu)K_0^{1-\alpha}\}^{1/(1-\beta)}$.) But if $\nu = 0$ (no technical progress), $\bar{C}$ simplifies to the Solow (1974) constant consumption path $C(t) = (1-\beta)\{K_0^{\alpha-\beta}[(\alpha-\beta)S_0]^{\beta}\}$, and only then are all five income measures defined, constant, and equal to consumption.

An idea of how big differences among income measures can be gained from a numerical example. If $\rho = 2$, $\alpha = 0.6$, $\beta = 0.05$, $\nu = 0.4$, $K_0 = 1000$, $S_0 = 100$ and time is measured in years, then to 3 decimal places, $\xi = 1.053$, $\sigma = 2.368$ and $\theta = 0.010$. The various instantaneous, annual rates in the economy at time $t = 0$ are

utility discount rate	$\rho\theta$	= 0.019	)	[18]
technical progress rate	$\nu\theta$	= 0.004	)	
consumption growth rate	$\sigma\theta$	= 0.023	)	
current interest rate	$(\xi+\sigma)\theta$	= 0.033	)	

These *initial* rates are the same order of magnitude as the constant rates used by Weitzman (1997) and other authors, and so are not wildly implausible. Inserting the numbers into [11]-[14] − and adding a calculation of sustainable income Y_m done by numerical simulation just for time zero, which we discuss later − then shows that the income measures vary dramatically in this example, being (to one decimal place):

welfare-equivalent income	$A(t)$	$= 1573.6\ C(t)$	)	[19]
wealth-equivalent income	$Y_e(t)$	$= 46.0\ C(t)$	)	
Sefton-Weale income	$SW(t)$	$= 3.3\ C(t)$	)	
sustainable income	$Y_m(0)$	$= 3.1\ C(0)$	)	
NNP	$Y(t)$	$= 2.3\ C(t)$	)	

The fact that $\alpha/\beta > \rho$ and $Y_m(0) > Y(0)$ here suggests that result (c) above may also apply to the case of positive technical progress.

However, any empirical significance of these results is hard to judge, since the rates all decline over time as $1/(1+\theta t)$ from those in [18], contrary to empirical experience in Western economies over the last two centuries or so. Perhaps more significant are results from an exact solution of the Stiglitz (1974) economy, where it can be shown[1] that for the parameter values $\rho = 0.025$, $\alpha = 0.6$, $\beta = 0.05$ and $\nu = 0.01$ (a fairly standard set of exponential rates, except for the role of α in $U(C)$), the asymptotic income measures are $A = 3.2C$, $Y_e = SW = 2.5C$ and $Y = 1.5C$.

3.4 Sustained growth

Another feature that could have been listed in the previous subsection, but deserves more prominence, is that optimal consumption in the hyperbolic economy is steadily growing if the discount rate is low enough ($\rho < (\alpha+\nu)/\beta \Rightarrow \sigma = \dot{C}/C > 0$). Moreover, such sustained growth can be optimal even

1. The formulae for the corresponding special case of Stiglitz's economy are $F = K^\alpha R^\beta e^{\nu t}$, $\phi = e^{-\rho t}$, $\zeta := (\rho-\tau)/(1-\beta)$, $\omega := (\tau-\beta\rho)/(1-\alpha)(1-\beta)$, $(\zeta+\omega)K_0^{1-\alpha} = \alpha\zeta^\beta S_0^\beta$ (the parameter restriction needed to start on an analytic path), $C = (\rho K_0/\alpha)e^{\omega t}$, $A = [1+(1-\alpha)\omega/\zeta]^{1/(1-\alpha)}C$, $Y_e = SW = (1+\omega/\zeta)C$, and $Y = (1-\tau/\rho)(1+\omega/\zeta)C$.

if there is no technical progress (i.e. if $\rho < \alpha/\beta$ and $\nu = 0$). This reflects how a hyperbolic utility discount rate declines over time, in a way that can match the declining return to capital in an economy with a stock of human-made capital, a stock of non-renewable resource, and no technical progress. By contrast, in the seminal example of such an economy in Dasgupta and Heal (1974), the discount rate is constant, and ultimately becomes greater than the declining return to capital. Hence optimal consumption asymptotically falls toward zero there, no matter how small the discount rate.

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We now discuss the three topics noted earlier: calculating sustainable income when there is exogenous technical progress; whether there is a "best" measure of income; and the motivation and time-consistency of the optimal path.

3.5 Sustainable income and exogenous technical progress

The estimated number for initial sustainable income in a numerical example of the hyperbolic economy, given in [19], was calculated using the following method. The method works for a production function $F(K,R,t) = K^{\alpha}R^{\beta}\pi(t)$ with a general progess factor $\pi(t)$, which includes both the exponential case considered by Solow (1974), $\pi(t) = e^{\nu t}$, and the hyperbolic case, $\pi(t) = (1+\theta t)^{\nu}$. Standard optimal control techniques give the Hotelling rule that an economy with sustainable income (maximum constant consumption) C_m, must follow, like any dynamically efficient economy:

$$\dot{F}_R/F_R = F_K$$
$$\Rightarrow \quad \alpha\dot{K}/K - (1-\beta)\dot{R}/R + \dot{\pi}/\pi = \alpha F/K$$
$$\Rightarrow \quad \dot{R}/R = (\dot{\pi}/\pi - \alpha C_m/K) \,/\, (1-\beta) \qquad [20]$$

Finding C_m numerically involves initially guessing a C_m and an initial resource depletion rate $R(0)$, integrating forward [20] and $\dot{K} = F - C_m$, and then iterating the initial guesses so that C_m is maximised while the initial resource stock S_0 is completely depleted at an 'infinite time horizon'. Such a horizon can of course never be approached in simulation, but this procedure appears to give stable estimates of C_m even when the termination time is greatly increased. A point of interest is that, contrary to Solow's (1974, p41) speculation, capital K does not approach zero on a sustainable income path, but grows without bound (which also happens in the special case in Section 3.3(d) where an analytic solution exists). This is because supplying resource flow $R(t)$ over an infinite time from a finite stock S_0 requires that R declines to zero from above, hence that $\dot{R}/R$ declines to zero from below. This is impossible in [20] if K approaches zero, since $\alpha C_m/K$ and hence $-\dot{R}/R$ will then grow without bound.[2]

3.6 *Which measure of income should be used?*

During the development of this paper, a number of commenters have remarked that it is unsatisfactory to give five, quantitatively quite different measures of income, and yet no reason to prefer one measure to another. However, the history of economic debate about income shows that any criterion, purporting to judge the relative merits of different income measures on a common scale, is bound to be disputed. This paper avoids the debate, by taking the alternative view that *income is not a well-enough defined concept for there to exist a universally accepted, exact measure of income*

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2. What happens in simulations, in either the exponential or hyperbolic case, is that R at first falls because K is at first relatively small. But then as K grows, $\alpha C_m/K$ falls below $\dot{\pi}/\pi$, and from [20] R grows again. C_m and $R(0)$ need to be iterated so that R is constant at the same moment as the resource stock is exhausted.

that is 'best'. This is fundamentally because measuring income can serve many different purposes, for example:

> "...charting business cycles, comparing prosperity among nations, observing industrial structure, measuring factor shares and so on. ...real income may be interpreted as a family of concepts, each member of which is best for some particular purpose." (Usher 1994, p124)

Along similar lines, Asheim (2000) noted that comparing prosperity among nations is a quite different task from measuring one nation's sustainability, and requires a different income measure; and that Hicks (1946, Ch 14) himself emphasised both sustainable income (our Y_m) and wealth-equivalent income (our Y_e) as valid income concepts. However, Hicks used a framework (a person facing exogenous prices, rather than a closed economy facing endogenous prices, as above) where these two income definitions are indistinguishable. So in particular, the phrase "Hicksian income" (used by Nordhaus 2000 and numerous other recent writers) is almost always contentious or ambiguous (see for example Vincent 2000, footnote 2), and has been deliberately avoided here.

The algebraic results above demonstrate that even as a measure of prosperity, income is hard to define uniquely. Clearly, a measure of current prosperity should take proper account of the future, and consumption alone is not a proper measure. But this leaves undefined what kind of future society wants, and exactly how to take account of it. There are many unresolved arguments about how one should choose from an infinitude of intertemporal welfare objectives, each of which leads to a different future with different accounting prices. Even when present value maximisation with a particular discount factor is chosen, there is still a difference, given a diminishing marginal utility of consumption, between the welfare-equivalence and wealth-equivalence methods of accounting for the future.

3.7 What is the economy's motivation, and is the optimal path time-consistent?

Two other questions raised during this paper's development are about the discount factor $\phi(t) = (1+\theta t)^{-\rho}$ that is a defining characteristic of the hyperbolic economy. First, is there a primitive welfare criterion underlying its use, and thus some explanation of what basic principles motivate the economy to follow the calculated optimal path? Second, does its use cause the optimal path to be time-inconsistent?

There is no easy answer to the first question. No elegant axiomatic foundation, of the kind that Koopmans (1960) established for exponential discounting, exists for hyperbolic discounting, and one may never exist. But even if it did, it would not necessarily resolve the choice between hyperbolic discounting and other criteria (while clear axiomatically, the choice between exponential discounting and maximin consumption remains unresolved). Meanwhile, the above results on income measurement still seem useful.

The second comment, about time consistency, can be answered more satisfactorily. The optimal consumption path in [7] *is* time-consistent, *if* one interprets the discount factor in a particular way. Consider a 're-optimisation' at some time $t = x \geq 0$ after the start of the optimal path, and use a redefined time variable starting from this time, $s := t-x$. Provided that the discount factor $\phi_x(s)$ to be used from $s = 0$ onwards, with $\phi(0) = 1$, is

$$\left.\begin{aligned} \phi_x(s) &= (1+\theta_x s)^{-\rho} \quad \text{with} \\ \theta_x &:= \{[\alpha(\xi-1)^\beta S_0^\beta / (\xi+\sigma)K_0^{1-\alpha}]^{-1/(1-\beta)} + x\}^{-1}, \end{aligned}\right) \qquad [21]$$

but not $:= \{\alpha(\xi-1)^\beta[S(0)]^\beta / (\xi+\sigma)[K(0)]^{1-\alpha}\}^{1/(1-\beta)}$ [22]

(where $S(0)$ and $K(0)$ are evaluated at $s = 0$, not $t = 0$)

or $:= \{\alpha(\xi-1)^\beta S_0^\beta / (\xi+\sigma)K_0^{1-\alpha}\}^{1/(1-\beta)}$ (= θ in [6]), [23] then $(1+\theta_x s)^{-\rho} = \{[1+\theta(x+s)] / (1+\theta x)\}^{-\rho}$,

and so $-\dot{\phi}_x(s)/\phi_x(s)\,|_{s=0} = -\dot{\phi}(t)/\phi(t)\,|_{t=x} = \rho\theta/(1+\theta x)$. [24]

[24] means that the instantaneous discount rate remains unchanged by the reoptimisation at $s = 0$ (i.e. $t = x$); and since x is arbitrary, the optimal path is thereby time-consistent. This consistency is achieved by abandoning Strotz's requirement that the discount factor $\phi(t_1,t_2)$, used to make utility at time t_2 comparable with an earlier time t_1, should depend on just the time lapse t_2-t_1 and purely psychological parameters (as noted by Asheim 2000, p31). In [21], ϕ is allowed to depend on the state of the economy at a fixed moment (specifically, on the initial capital and resource stocks, K_0 and S_0) and on the absolute time since that moment. Neither of the formulae [22] or [23] for reoptimisation includes absolute time, and because of this either would cause time-inconsistency. As long as one accepts the idea that people's discounting can be affected by the absolute state of the economy's stocks and technology (and hence absolute time), there is no reason to prefer [22] or [23] to [21].

4. Conclusions

Exact solutions have been presented for the optimal path of a 'hyperbolic' theoretical economy with human-made capital, a non-renewable resource, exogenous technical progress, and specific functional forms. This economy illustrates some significant points in recent literature on income and sustainability accounting, and may prove useful as a testbed for future theoretical enquiry. Hyperbolic discounting gives the ethically attractive property that optimal consumption grows forever if the discount rate is low enough, even if there is no exogenous technical progress. This avoids some well known problems of the Solow (1974) constant consumption path, which is a special case of the hyperbolic economy with a particular discount rate and zero technical progress. Also in the hyperbolic economy, five measures of income − welfare-equivalent income, wealth-equivalent income, Sefton-Weale income, net national product (NNP) and sustainable income − are all distinct theoretically, with the first four measures in descending size order, and with quite different values in a plausible numerical example. So it is hard to view any one definition of income as 'correct', Hicksian or otherwise. Instead, one is forced to recognise that different measures of 'income' serve different purposes. Time-consistency is not a problem, as long as the discount parameter is defined in terms of the original stocks and absolute time in such a way that reoptimising at a later time continues with the original hyperbolic discount factor. Further research on any empirical significance of the above results for both economies would seem worthwhile.

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REFEREES' NOTE FOR "EXACT MEASURES OF INCOME IN A HYPERBOLIC ECONOMY" (23 January 2002 draft)

These are the calculations implied by the note at the end of Section 1 that "full details are available from the author", and they would be placed on my website. But if the referees prefer, they could be edited down to form an Appendix included in the paper. (The division into a preliminary section and then Appendix 1 is there because originally the paper had another section which required an Appendix 2.)

In the hyperbolic economy the maximisation problem is

$$\max_{C,R} \int_0^\infty \phi(t)U[C(t)]dt \quad \text{s.t. } \dot{K} = J[F(K,R,t),C], \quad \dot{S} = -R; \quad K(0) = K_0,\ S(0) = S_0 \qquad \text{[A0.1]}$$

and the current value Hamiltonian is

$$H = U(C) + \mu^K\dot{K} + \mu^S\dot{S} = U(C) + \mu^K J[F(K,R,t),C] - \mu^S R \qquad \text{[A0.2]}$$

The necessary first order conditions for an interior solution are

$$H_C = U_C + \mu^K J_C = 0 \quad \Rightarrow \quad \mu^K = -U_C/J_C \qquad \text{[A0.3]}$$

$$H_K = \mu^K J_F F_K = -(\dot{\phi}/\phi)\mu^K - \dot{\mu}^K \quad \Rightarrow \quad \dot{\mu}^K/\mu^K = -(\dot{\phi}/\phi) - J_F F_K \qquad \text{[A0.4]}$$

$$H_R = \mu^K J_F F_R - \mu^S = 0 \quad \Rightarrow \quad \mu^S = \mu^K J_F F_R \qquad \text{[A0.5]}$$

$$H_S = 0 = -(\dot{\phi}/\phi)\mu^S - \dot{\mu}^S \quad \Rightarrow \quad \dot{\mu}^S/\mu^S = -(\dot{\phi}/\phi) \qquad \text{[A0.6]}$$

Appendix 1. Optimal solution paths for the hyperbolic economy

In [1], $U(C) = C^{1-\alpha}/(1-\alpha), \qquad \phi(t) = (1+\theta t)^{-\rho}$ [A1.1]

$\dot{K} = J(F,C) = F - C \qquad F(K,R,t) = K^\alpha R^\beta (1+\theta t)^\nu$ [A1.2]

[A0.3-6] then give

$$U_C = C^{-\alpha} = \mu^K; \qquad \mu^S = \mu^K F_R = C^{-\alpha}\beta K^\alpha R^{\beta-1}(1+\theta t)^\nu$$

$$-\alpha\dot{C}/C = -\dot{\phi}/\phi - F_K = \rho\theta/(1+\theta t) - \alpha K^{\alpha-1}R^\beta(1+\theta t)^\nu \qquad \text{[A1.3]}$$

$$\dot{F}_R/F_R = \alpha\dot{K}/K - (1-\beta)\dot{R}/R + \theta\nu/(1+\theta t) = F_K = \alpha F/K$$

$$\Rightarrow \theta\nu/(1+\theta t) - (1-\beta)\dot{R}/R = \alpha C/K \qquad \text{[A1.4]}$$

Also $\delta_\infty := 1 \,/ \int_t^\infty [\phi(s)/\phi(t)]ds = (1+\theta t)^{-\rho} \,/\, [(1+\theta s)^{-\rho+1}/(1-\rho)\theta]_t^\infty$

$$= (\rho-1)\theta/(1+\theta t). \qquad \text{[A1.5]}$$

Seek "hyperbolic steady state" solution

$$C = C_0(1+\theta t)^{\sigma},\ K = K_0(1+\theta t)^{\sigma+1} \quad \text{[A1.6]}$$

$$S = S_0(1+\theta t)^{-(\xi-1)},\ R = (\xi-1)\theta S_0(1+\theta t)^{-\xi} \quad \text{[A1.7]}$$

Comparing rates of growth and constant terms in [A1.3-4] then gives

$$\sigma = (\sigma+1)\alpha - \xi\beta + \nu \quad \text{[A1.8]}$$

$$C_0 = K_0^{\alpha}[(\xi-1)\theta S_0]^{\beta} - (\sigma+1)\theta K_0 \quad \text{[A1.9]}$$

$$\alpha\sigma\theta = \alpha K_0^{\alpha-1}[(\xi-1)\theta S_0]^{\beta} - \rho\theta \quad \text{[A1.10]}$$

and $$C_0 = (K_0/\alpha)[\theta\nu+(1-\beta)\xi\theta] \quad \text{[A1.11]}$$

[A1.9-11] $\Rightarrow$ $$C_0/K_0 = [\nu+(1-\beta)\xi]\theta/\alpha \quad \text{[A1.12]}$$

$$= K_0^{\alpha-1}[(\xi-1)\theta S_0]^{\beta} - (\sigma+1)\theta = (\alpha\sigma+\rho)\theta/\alpha - (\sigma+1)\theta$$

$$\Rightarrow \quad \nu+(1-\beta)\xi = \rho-\alpha \quad \Rightarrow \quad \xi = (\rho-\alpha-\nu)/(1-\beta) \quad \text{[4]}$$

[A1.8-12],[4] $\Rightarrow$ $$\sigma = (\alpha+\nu-\beta\xi)/(1-\alpha) = (\alpha+\nu-\beta\rho)/(1-\alpha)(1-\beta) \quad \text{[5]}$$

and $$\alpha\sigma+\rho = \xi+\sigma = [\rho(1-\alpha-\beta)+\alpha(\alpha+\nu)] / (1-\alpha)(1-\beta) \quad \text{[A1.13]}$$

Non-renewable resource stock requires $\dot{S} < 0$, hence $\xi-1 > 0$, hence

$$\rho > 1+\alpha-\beta+\nu. \quad \text{[3]}$$

[A1.10] $\Rightarrow$ $\theta^{1-\beta} = \alpha K_0^{\alpha-1}[(\xi-1)S_0]^{\beta}/(\alpha\sigma+\rho)$. With [A1.13], [4] this gives

$$\theta = [\alpha(\xi-1)^{\beta}S_0^{\beta}/(\xi+\sigma)K_0^{1-\alpha}]^{1/(1-\beta)} \quad \text{[6]}$$

Hence K, S, R are as in [8-9]; and [A1.12], [4] give

$$C = [(\rho-\alpha)\theta K_0/\alpha]\,(1+\theta t)^{\sigma}. \quad \text{[7]}$$

$F = K_0^{\alpha}\,[(\xi-1)\theta S_0]^{\beta}\,(1+\theta t)^{\sigma} = K_0^{\alpha}\,[(\xi-1)\theta S_0]^{\beta}\,(1+\theta t)^{\sigma}$, and using [6]

$$= K_0^{\alpha}\,[\theta(\xi+\sigma)K_0^{1-\alpha}/\alpha]\,(1+\theta t)^{\sigma} = [(\xi+\sigma)/(\rho-\alpha)]C$$

$$F_R = \beta F/R = [\beta(\xi+\sigma)\theta K_0/\alpha]\,(1+\theta t)^{\sigma} / (\xi-1)\theta S_0(1+\theta t)^{-\xi}$$

$$= [\beta K_0(\xi+\sigma)/(\xi-1)\alpha S_0](1+\theta t)^{\xi+\sigma}$$

$$F_R R = \beta F = [\beta\xi/(\rho-\alpha)](1+\sigma/\xi)C$$

$$\Phi = \phi U_C = (1+\theta t)^{-\rho}\,[(\rho-\alpha)\theta K_0/\alpha]^{-\alpha}\,(1+\theta t)^{-\alpha\sigma}$$

$$\Rightarrow \quad r = -\dot{\Phi}/\Phi = (\rho+\alpha\sigma)\theta/(1+\theta t) = (\xi+\sigma)\theta/(1+\theta t) \quad \text{[10]}$$

$r_{\infty} = (\xi+\sigma-1)\theta/(1+\theta t)$ follows analogously to δ_{∞}. [A1.14]

$$U = C_0^{1-\alpha}(1+\theta t)^{\sigma(1-\alpha)}/(1-\alpha) = C_0^{1-\alpha}(1+\theta t)^{(\alpha+\nu-\beta\rho)/(1-\beta)}/(1-\alpha)$$

$$\phi U = (1+\theta t)^{-\rho}C_0^{1-\alpha}(1+\theta t)^{[(\alpha+\nu-\beta\rho)/(1-\beta)]}/(1-\alpha) = C_0^{1-\alpha}(1+\theta t)^{-\xi}/(1-\alpha)$$

$$W = (1+\theta t)^{\rho}C_0^{1-\alpha}(1+\theta t)^{-\xi+1}/(1-\alpha)(\xi-1)\theta = (1+\theta t)U/\theta(\xi-1)$$

$$\Rightarrow \delta_\infty W \quad = [(\rho-1)\theta/(1+\theta t)](1+\theta t)U/\theta(\xi-1) = (\rho-1)U/(\xi-1)$$

$$= [1+(\rho-\xi)/(\xi-1)]U = [1+(1-\alpha)\sigma/(\xi-1)]U$$

$$\dot{K} - F_R R = (\sigma+1)\alpha C/(\rho-\alpha) - \beta(\xi+\sigma)C/(\rho-\alpha)$$

$$= [(\sigma+1)\alpha\xi - \beta\xi(\xi+\sigma) + \nu(\xi+\sigma) - \nu(\xi+\sigma)]\ C/\xi(\rho-\alpha)$$

$$= \{\ (\alpha+\nu-\beta\rho+1-\alpha-\beta+\alpha\beta)\alpha(\rho-\alpha-\nu) +$$

$$(\nu-\beta\nu-\beta\rho+\alpha\beta+\beta\nu)[\rho(1-\alpha-\beta)+\alpha(\alpha+\nu)] - \nu(\xi+\sigma)(1-\alpha)(1-\beta)^2\ \}$$

$$\times\ C/\xi(\rho-\alpha)(1-\alpha)(1-\beta)^2$$

$$= \{(\nu-\beta\rho+\alpha\beta)\rho(1-\beta) + (1-\beta)\alpha(\rho-\alpha-\nu) - \nu(\xi+\sigma)(1-\alpha)(1-\beta)^2\}$$

$$\times\ C/\xi(\rho-\alpha)(1-\alpha)(1-\beta)^2$$

$$= [\sigma-\nu(\xi+\sigma)/(\rho-\alpha)]C/\xi$$

$$\Theta \quad = \int_t^\infty [\Phi(s)/\Phi(t)]C(s)ds = (1+\theta t)^{(\xi+\sigma)}\ [\int_t^\infty (1+\theta s)^{-(\xi+\sigma)}C_0(1+\theta s)^{\sigma}ds$$

$$= (1+\theta t)C/\theta(\xi-1)$$

$$A = \{(1-\alpha)[1+(1-\alpha)\sigma/(\xi-1)]U\}^{1/(1-\alpha)} = [1+(1-\alpha)\sigma/(\xi-1)]^{1/(1-\alpha)}C \qquad [11]$$

$$Y_e = r_\infty\Theta = [(\xi+\sigma-1)\theta/(1+\theta t)](1+\theta t)C/\theta(\xi-1) = [1+\sigma/(\xi-1)]C \qquad [12]$$

S-W income $= \int_t^\infty [r(s)\Phi(s)C(s)/\Phi(t)]ds$

$$= (1+\theta t)^{\xi+\sigma} \int_t^\infty (\xi+\sigma)\theta(1+\theta s)^{-1-(\xi+\sigma)+\sigma}C_0 ds$$

$$= (1+\theta t)^{\xi+\sigma} \int_t^\infty (\xi+\sigma)\theta(1+\theta s)^{-1-\xi}C_0 ds = [(\xi+\sigma)/\xi]C = (1 + \sigma/\xi)C \qquad [13]$$

$$Y \quad = C + \dot{K} - F_R R$$

$$= [\xi+\sigma-\nu(\xi+\sigma)/(\rho-\alpha)]C/\xi = [1-\nu/(\rho-\alpha)](1+\sigma/\xi)C \qquad [14]$$

Sustainable income ($\alpha > \beta$, and no technical progress, $\nu = 0$ case, only)

From Solow (1974, p39), sustainable income when $\nu = 0$ is:

$Y_m(t) \quad = (1-\beta)\{[K(t)]^{\alpha-\beta}[(\alpha-\beta)S(t)]^{\beta}\}^{1/(1-\beta)}$

$\quad = (1-\beta)\{K_0^{\alpha-\beta}[(\alpha-\beta)S_0]^{\beta}\}^{1/(1-\beta)}(1+\theta t)^{x}$

where from [8-9], $x = [(\sigma+1)(\alpha-\beta)-(\xi-1)\beta] / (1-\beta)$

$= [(\alpha\text{-}\beta\rho+1\text{-}\alpha\text{-}\beta+\alpha\beta)(\alpha\text{-}\beta)/(1\text{-}\alpha)(1\text{-}\beta) - (\rho\text{-}\alpha\text{-}1+\beta)\beta/(1\text{-}\beta)] / (1\text{-}\beta)$

$= [(-\beta\rho+1-\beta+\alpha\beta)(\alpha-\beta) - (\rho-\alpha-1+\beta)\beta(1-\alpha)] / (1-\alpha)(1-\beta)^2$

$= [(-\beta\rho+1-\beta+\alpha\beta+\beta-\alpha\beta)(\alpha-\beta) - (\rho-1)\beta(1-\alpha)] / (1-\alpha)(1-\beta)^2$

$= [(1-\beta\rho)(\alpha-\beta) - (\beta\rho-\beta)(1-\alpha)] / (1-\alpha)(1-\beta)^2$

$= [\alpha-\alpha\beta\rho-\beta+\beta^2\rho-\beta\rho+\beta+\alpha\beta\rho-\alpha\beta)] / (1-\alpha)(1-\beta)^2$

$= [\alpha+\beta^2\rho-\beta\rho-\alpha\beta)] / (1-\alpha)(1-\beta)^2$

$= (\alpha-\beta\rho) / (1-\alpha)(1-\beta)$

$= \sigma$, i.e. Y_m has the same time dependence as the other income measures

And from [6] and [7],

$C_0 = (\rho-\alpha)\ [\alpha(\xi-1)^{\beta}S_0^{\beta}K_0^{1-\beta}/(\xi+\sigma)K_0^{1-\alpha}]^{1/(1-\beta)} / \alpha$

$\Rightarrow\ [\alpha^{\beta}(\xi-1)^{\beta}S_0^{\beta}K_0^{\alpha-\beta}/(\xi+\sigma)]^{1/(1-\beta)} = C_0/(\rho-\alpha)$

$\Rightarrow\ (1-\beta)[S_0^{\beta}K_0^{\alpha-\beta}]^{1/(1-\beta)} = (1-\beta)C_0\ [(\xi+\sigma)/(\xi-1)^{\beta}\alpha^{\beta}]^{1/(1-\beta)} / (\rho-\alpha)$

$\Rightarrow (1\text{-}\beta)\{K_0^{\alpha\text{-}\beta}[(\alpha\text{-}\beta)S_0]^{\beta}\}^{1/(1\text{-}\beta)} = (1\text{-}\beta)C_0[(\xi+\sigma)(\alpha\text{-}\beta)^{\beta}/(\xi\text{-}1)^{\beta}\alpha^{\beta}]^{1/(1-\beta)} / (\rho\text{-}\alpha)$

Multiplying both sides by $(1+\theta t)^{\sigma}$ then gives

$\Rightarrow\ Y_m(t) = (1-\beta)C(t)[(\xi+\sigma)(\alpha-\beta)^{\beta}/(\xi-1)^{\beta}\alpha^{\beta}]^{1/(1-\beta)} / (\rho-\alpha)$

$\quad = C(t)[(\xi+\sigma)(\alpha-\beta)^{\beta}/(\xi-1)^{\beta}\alpha^{\beta}]^{1/(1-\beta)} / \xi$

$\quad = C(t)[(1+\sigma/\xi)(\alpha-\beta)^{\beta}/(1-1/\xi)^{\beta}\alpha^{\beta}]^{1/(1-\beta)}$

$\quad = C(t)[(1+\sigma/\xi)^{\beta}(\alpha-\beta)^{\beta}/(1-1/\xi)^{\beta}\alpha^{\beta}]^{1/(1-\beta)}(1+\sigma/\xi)$

$$Y_m(t) = [(\xi+\sigma)(\alpha-\beta)/(\xi-1)\alpha]^{\beta/(1-\beta)}(1+\sigma/\xi)\ C(t) \qquad [15]$$

In which case

$Y_m/Y = [(\xi+\sigma)(\alpha-\beta)/(\xi-1)\alpha]^{\beta/(1-\beta)}$

$\Rightarrow (Y_m/Y)^{(1-\beta)/\beta} \quad = (\xi+\sigma)(\alpha-\beta) / (\xi-1)\alpha$

$\quad = [\rho(1-\alpha-\beta)+\alpha^2](\alpha-\beta) / (1-\alpha)(\rho-\alpha-1+\beta)\alpha$

$\quad = [\rho(\alpha-\alpha^2-\alpha\beta-\beta+\alpha\beta+\beta^2)+\alpha^2(\alpha-\beta)] / (\alpha-\alpha^2)(\rho-\alpha-1+\beta)$

$\quad = [\rho(\alpha-\beta+\beta^2)+\alpha^2(\alpha-\beta-\rho)] / [\alpha(\rho-1+\beta)+\alpha^2(\alpha-\beta-\rho)]$

so $Y_m > Y$ if $\quad \rho(\alpha-\beta+\beta^2) > \alpha(\rho-1+\beta)$

i.e. if $\quad -\beta\rho+\beta^2\rho > -\alpha+\alpha\beta$

i.e. if $\quad \alpha/\beta > \rho.$ [16]

The constant consumption case

From [4] and [5], $\alpha+\nu-\beta\rho = 0 \;\Rightarrow\; \xi = \rho$ and $\sigma = 0$.

Hence from [6], $\theta = [\alpha(\rho-1)^\beta S_0{}^\beta/\rho K_0{}^{1-\alpha}]^{1/(1-\beta)}$

Hence from [7], $\bar{C} = (\rho-\alpha)[\alpha(\rho-1)^\beta S_0{}^\beta/\rho K_0{}^{1-\alpha}]^{1/(1-\beta)}K_0/\alpha$

$= (\rho-\alpha)\{K_0{}^{\alpha-\beta}[\alpha(\rho-1)S_0]^\beta/\rho\}^{1/(1-\beta)}$

Then substituting $\rho = (\alpha+\nu)/\beta$, $\rho-\alpha = [\alpha(1-\beta)+\nu]/\beta$, $\rho-1 = (\alpha+\nu-\beta)/\beta$ gives

$\bar{C} = \{[\alpha(1-\beta)+\nu]/\beta\}\{K_0{}^{\alpha-\beta}[\alpha(\alpha+\nu-\beta)S_0/\beta]^\beta/[(\alpha+\nu)/\beta]\}^{1/(1-\beta)}$

$= [\alpha(1-\beta)+\nu]\{K_0{}^{\alpha-\beta}[\alpha(\alpha+\nu-\beta)S_0]^\beta/(\alpha+\nu)\}^{1/(1-\beta)}$ [17]

and also gives

$\theta = \{\alpha[(\alpha+\nu-\beta)/\beta]^\beta S_0{}^\beta/[(\alpha+\nu)/\beta]K_0{}^{1-\alpha}\}^{1/(1-\beta)}$

$= \{\alpha^{1-\beta}\beta^{1-\beta}[\alpha(\alpha+\nu-\beta)]^\beta S_0{}^\beta/(\alpha+\nu)K_0{}^{1-\alpha}\}^{1/(1-\beta)}$

$= \alpha\beta\{[\alpha(\alpha+\nu-\beta)]^\beta S_0{}^\beta/(\alpha+\nu)K_0{}^{1-\alpha}\}^{1/(1-\beta)}$ as given.

[ends]